NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1311

CONTRIBUTIONS TO THE THEORY OF THE SPREADING

OF A FREE JET ISSUING FROM A NOZZLE

By W. Szablewski

Translation of "Zur Theorie der Ausbreitung eines aus einer Düse austretenden freien Strahls." Untersuchungen und Mitteilungen Nr. 8003, September 1944.

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CONTRIBUTIONS TO THE THEORY OF THE SPREADING OF A FREE JET ISSUING FROM A NOZZLE*

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PART I.- THE FLOW FIELD IN THE CORE REGION

ABSTRACT:

For the flow field of a free jet leaving a nozzle of circular cross section in a medium with straight uniform flow field, approximate formulas are presented for the calculation of the velocity distribution and the dimensions of the core region. The agreement with measured results is satisfactory.

OUTLINE:

I. INTRODUCTION AND SURVEY OF METHOD AND RESULTS

Knowledge of the flow field of a free jet leaving a nozzle is of basic importance for practical application.

Investigation of such a flow field is a problem of free turbulence.

In theoretical research the following specialized cases of our problem have already been treated:

(a) The mixing of two plane jets, the so-called plane jet boundary. These conditions are encountered in the immediate proximity of the nozzle.

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*"Zur Theorie der Ausbreitung eines aus einer Düse austretenden freien Strahls." Untersuchungen und Mitteilungen Nr. 8003, September 1944.

(b) The spreading of a rotationally-symmetrical jet issuing from a point-shaped slot in a wall, the so-called rotationally-symmetrical jet spreading. This state defines the conditions at very large distance from the nozzle.

In considering a free jet leaving a nozzle of circular cross section, we may subdivide the spreading procedure, according to an essential characteristic, into two different regions:

(1) Region where a zone of undiminished velocity is still present (the so-called jet core). We shall call this range, which extends from the nozzle to the core end, the core region. For the immediate proximity of the nozzle the conditions of the plane jet boundary exist.

(2) The region of transition adjoining the core region which is characterized by a constant decrease of the central velocity. This region opens into the region of the rotationally symmetrical jet spreading mentioned above.

So far, there exists only an investigation concerning the core region (reference 1); it is limited to the case where the surrounding medium is in a state of rest.

Method and Results

In the present paper, the spreading of a jet in the core region is treated for the general case where the surrounding medium has a straight uniform flow field (or, respectively, where the nozzle from which the jet issues moves at a certain velocity through the surrounding medium at rest).

The theoretical investigation is based (reference 2) on the more recent Prandtl expression for the momentum transport

$$\epsilon(x) = \kappa b(x) \left| \bar{u}_{max} - \bar{u}_{min} \right|$$

One then obtains in the rotationally symmetrical case the following equations:

Continuity:

$$\frac{\partial(r\bar{u})}{\partial x} + \frac{\partial(r\bar{v})}{\partial r} = 0$$

Momentum transport:

$$\bar{u}\,\frac{\partial \bar{u}}{\partial x} + \bar{v}\,\frac{\partial \bar{u}}{\partial r} = \epsilon\left(\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\,\frac{\partial \bar{u}}{\partial r}\right)$$

where

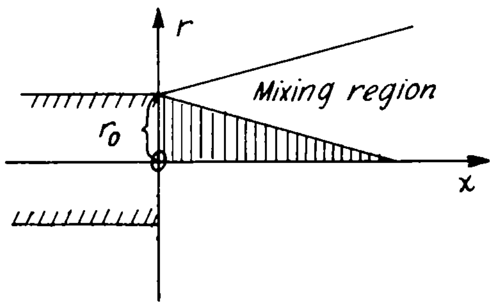
$$\epsilon = \kappa b(x)(u_1 - u_0)$$

u_1 = velocity at the jet core

u_0 = velocity of the surrounding medium

$$\left(u_1 > u_0\right)$$

With reference to the present problem



we introduce, instead of r,

$$\eta = \frac{r - r_0}{x}$$

as independent variable. We obtain:

Continuity

$$\left(\eta + \frac{r_0}{x}\right)\left(x\,\frac{\partial \bar{u}}{\partial x} - \eta\,\frac{\partial \bar{u}}{\partial \eta}\right) + \bar{v} + \left(\eta + \frac{r_0}{x}\right)\frac{\partial \bar{v}}{\partial \eta} = 0$$

Momentum transport

$$\bar{u}\left(x \frac{\partial \bar{u}}{\partial x} - \eta \frac{\partial \bar{u}}{\partial \eta}\right) + \bar{v} \frac{\partial \bar{u}}{\partial \eta} = \frac{\epsilon(x)}{x}\left(\frac{\partial^2 \bar{u}}{\partial \eta^2} + \frac{1}{\eta + \frac{r_0}{x}} \frac{\partial \bar{u}}{\partial \eta}\right)$$

<u>Velocity distribution in the Core Region</u>

We limit our considerations at first to small disturbances of the flow field; that is, to relatively small differences in velocity $\left(\frac{u_1 - u_0}{u_1} \text{ small quantity}\right)$. The partial differential equation for the momentum transport may then be linearized

$$r \frac{\partial \bar{u}}{\partial x} = \epsilon(x)\left(\frac{\partial \bar{u}}{\partial r} + r \frac{\partial^2 \bar{u}}{\partial r^2}\right)$$

where

$$\epsilon(x) = \kappa b(x) \frac{u_1 - u_0}{u_1}$$

(It should be noted that by the transformation $\mu = \frac{r}{\sqrt{\epsilon(x)}}$ this equation is transformed into the equation

$$\frac{\partial^2 \bar{u}}{\partial \mu^2} + \frac{\partial \bar{u}}{\partial \mu} \frac{1}{\mu} - \frac{\partial \bar{u}}{\partial x} = 0$$

which represents a heat conduction equation.) With $\eta = \frac{r - r_0}{x}$ instead of r one obtains from the equation of momentum

$$\frac{\partial^2 \bar{u}}{\partial \eta^2} + \frac{\partial \bar{u}}{\partial \eta}\left[\frac{1}{\eta + \frac{r_0}{x}} + \eta \frac{x}{\epsilon(x)}\right] - \frac{\partial \bar{u}}{\partial x} \frac{x^2}{\epsilon(x)} = 0$$

if

$$\eta + \frac{r_0}{x} \neq 0$$

This equation is a linear partial differential equation of second order of parabolic type. For the plane case $\left(\frac{r_0}{x} \longrightarrow \infty\right)$ this results in the equation

$$\frac{d^2\bar{u}}{d\eta^2} + \frac{d\bar{u}}{d\eta}\,\eta\left(\frac{x}{\epsilon(x)}\right)_{\frac{x}{r_0}=0} = 0$$

with

$$\left(\frac{x}{\epsilon(x)}\right)_{\frac{x}{r_0}=0} = \frac{1}{\kappa c\left(\frac{u_1 - u_0}{u_1}\right)}$$

therein $c = \lim\limits_{\frac{x}{r_0}=0} \frac{b(x)}{x}$ and is to be regarded as a function of $\frac{u_1 - u_0}{u_1}$.

With the boundary conditions taken into consideration, the integration yields

$$\frac{\bar{u}}{u_0} = \frac{1}{\sqrt{\pi}}\left(\frac{u_0}{u_1} - 1\right)\int_0^{\sigma_0\eta} e^{-(\sigma_0\eta)^2}\, d(\sigma_0\eta) + \frac{1}{2}\left(1 + \frac{u_0}{u_1}\right)$$

with

$$\sigma_0 = \frac{1}{\sqrt{2\kappa c\left(\frac{u_1 - u_0}{u_1}\right)}}$$

We now obtain an approximate solution of our problem by generalizing the plane velocity distribution and setting up the following formulation:

$$\frac{\bar{u}}{u_1} = \frac{1}{\sqrt{\pi}}\left(\frac{u_0}{u_1} - 1\right)\int_0^{\left[\sigma_1\eta+\sigma_2\right]} e^{-\left[\sigma_1\eta+\sigma_2\right]^2}\, d\left[\sigma_1\eta + \sigma_2\right] + \frac{1}{2}\left(1 + \frac{u_0}{u_1}\right)$$

With

$$\sigma_1(x) = \frac{1}{\sqrt{2\kappa\left(\frac{u_1 - u_0}{u_1}\right)\frac{b}{x}}}$$

$$\sigma_2(x) = \frac{1}{2}\sqrt{2\kappa\left(\frac{u_1 - u_0}{u_1}\right)}\,\frac{1}{\left(\frac{x}{r_0}\right)}\int_0^{\frac{x}{r_0}}\left(\frac{x}{r_0}\right)\sqrt{\frac{b}{x}}\;d\left(\frac{x}{r_0}\right)$$

we obtain a function which corresponds to the exact solution for small $\frac{x}{r_0}$ as well as for large positive η, thus in boundary zones of the region of integration as well as in the interior of the region along the jet $\eta = 0$.

If we now consider larger disturbances, the solutions obtained for small disturbances are to be regarded as a first approximation.

For the plane case the solution for arbitrary $\frac{u_1 - u_0}{u_1}$ already exists, compare Görtler (reference 3). It is found that, purely with respect to shape, even the first approximation represents a very good approximation. The velocity distribution calculated by Görtler still shows an uncertainty insofar as $\overline{u}(\eta + a)$, with $\overline{u}(\eta)$, also represents a solution. This uncertainty here may be eliminated, because for the jet core vanishing of the transverse component $\overline{v}$ is required. Therewith the initial profile of the velocity distribution for arbitrary $\frac{u_1 - u_0}{u_1}$ is then unequivocally fixed.

If we limit ourselves, with respect to shape, to the first approximation, the initial profile is

$$\frac{\overline{u}}{u_1} = \frac{1}{\sqrt{\pi}}\left(\frac{u_0}{u_1} - 1\right)\int_0^{\xi} e^{-\xi^2}\,d\xi + \frac{1}{2}\left(1 + \frac{u_0}{u_1}\right)$$

where

$$\xi = \sigma_0\eta - 0.36\left(\frac{u_1 - u_0}{u_1}\right)$$

and

$$\sigma_0 = \frac{1}{\sqrt{2\kappa c\left(\frac{u_1 - u_0}{u_1}\right)}}$$

For the further development of the profiles starting from this initial profile the regularity found for small disturbances is then taken as a basis

$$\frac{\bar{u}}{u_1} = \frac{1}{\sqrt{\pi}}\left(\frac{u_0}{u_1} - 1\right)\int_0^{\eta^*} e^{-\eta^{*2}}\, d\eta^* + \frac{1}{2}\left(1 + \frac{u_0}{u_1}\right)$$

where

$$\eta^* = \sigma_1\eta - 0.36\left(\frac{u_1 - u_0}{u_1}\right) + \sigma_2$$

For $\frac{u_1 - u_0}{u_1} \longrightarrow 0$, this function is transformed into the approximation function constructed for small disturbances. How far it may be considered an approximation in the region for arbitrary disturbance is not investigated in more detail.

The functions appearing in the integral $\sigma_1(x)$, $\sigma_2(x)$ result from the approximation calculation for the dimensions of the core region, carried out on the basis of the momentum theorem.

Calculation of the Transverse Component

The transverse component $\bar{v}$ of the flow is determined from the continuity equation

$$\bar{v} = -\frac{1}{r}\int^r \left(r\,\frac{\partial\bar{u}}{\partial x}\right)dr \qquad (r \neq 0)$$

or $\bar{v} = -\frac{1}{\left(\eta + \frac{r_0}{x}\right)}\int^{\eta}\left(\eta + \frac{r_0}{x}\right)\left(x\,\frac{\partial\bar{u}}{\partial x} - \eta\,\frac{\partial\bar{u}}{\partial\eta}\right)d\eta$ respectively, with our approximate function being substituted for $\bar{u}$.

The integration constant is determined from the requirement that at the jet core the transverse flow component vanishes.

In order to avoid complication of the calculation, rectilinear course of the mixing width $b(x)$ is assumed. This assumption proves approximately correct as results from the calculation of the dimensions of the core region.

Dimensions of the Core Region

The dimensions of the core region (jet core and width of the mixing zone) are calculated according to a formulation of the momentum theorem

$$\left(\bar{u} - u_0\right)\bar{v}r - \frac{\partial}{\partial x}\int_r^{\infty} \bar{u}\left(\bar{u} - u_0\right)r\ dr \qquad (= r\tau xy)$$

$$= r\left(\kappa b(x)\left(u_1 - u_0\right)\right)\frac{\partial u}{\partial r}$$

indicated by Tollmien (reference 4).

The occurring integrals as well as the $\frac{\partial \bar{u}}{\partial r}$ defining the shearing stress are determined approximately with the course of the velocity distribution assumed rectilinear

$$\frac{\bar{u} - u_0}{u_1} = \left(\frac{u_1 - u_0}{u_1}\right)(1 - \eta)$$

Then there result for the limiting curve $d(x)$ of the jet core and the width $b(x)$ of the mixing zone two ordinary differential equations of the first order which can be reduced to one equation

$$\frac{dy}{dx} = f(x) \qquad \left(y = \int f(x)\ dx\right)$$

This integral can be represented with the aid of elementary functions; however, for simplicity its calculation here is performed by graphical method.

κ appears as the only empirical constant which results by comparison with measurements given by Tollmien (reference 4) as $\underline{\kappa = 0.01576}$.

Comparison with Measurements

In order to carry through a comparison between theory and experiment, a measurement for the case $\frac{u_1 - u_0}{u_1} = 0.5$ was performed with the test arrangement described in reference 5.

The comparison with the theory offers satisfactory results if one takes into consideration that the effective radius of the nozzle flow referring to a rectangular velocity distribution is different from the geometrical radius.

II. CALCULATION OF THE FLOW FIELD

(a) Velocity Distribution in the Core Region

We base the theoretical investigation on the more recent Prandtl expression for the turbulent momentum transport

$$\epsilon(x) = \kappa b(x) \left| \bar{u}_{max} - \bar{u}_{min} \right|$$

where κ = dimensionless proportionality factor, b = measure for width of the mixing zone, and $\bar{u}$ = temporal mean value of the velocity.

We have at our disposal, for calculation of the flow field, the continuity equation and the momentum equation for the main direction of motion, which read in rotationally-symmetrical rotation

Continuity

$$\frac{\partial(r\bar{u})}{\partial x} + \frac{\partial(r\bar{v})}{\partial r} = 0$$

Momentum transport

$$\bar{u}\frac{\partial\bar{u}}{\partial x} + \bar{v}\frac{\partial\bar{u}}{\partial r} = \epsilon(x)\left(\frac{\partial^2\bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{u}}{\partial r}\right)$$

where

$$\epsilon(x) = \kappa b(x)\left(u_1 - u_0\right)$$

u_1 = velocity of the issuing jet

u_0 = straight uniform velocity of the surrounding medium

$$u_1 > u_0$$

We may integrate the continuity equation by introduction of a flow potential ψ

$$r\bar{u} = \frac{\partial \psi}{\partial r} \qquad r\bar{v} = -\frac{\partial \psi}{\partial x}$$

The momentum equation then is transformed into

$$\frac{\partial \psi}{\partial r}\frac{\partial^2 \psi}{\partial x\,\partial r} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial r} = \epsilon(x)\left(r\,\frac{\partial^3 \psi}{\partial r^2} - \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r}\right)$$

where $\epsilon(x) = \kappa b(x)\left(\frac{u_1 - u_0}{u_1}\right)$ if we make the velocity dimensionless by division by u_1. According to a method applied by Görtler (reference 3) we set up for ψ the expression

$$\psi = \psi_0 + \left(\frac{u_1 - u_0}{u_1}\right)\psi_1 + \left(\frac{u_1 - u_0}{u_1}\right)^2 \psi_2 + \ldots$$

developing ψ in powers of the parameter $\left(\frac{u_1 - u_0}{u_1}\right)$. Therein ψ_0 is the potential of an undisturbed flow $(u_1 = u_0)$; thus

$$\frac{\partial \psi_0}{\partial r} = ru_1 \qquad \frac{\partial \psi_0}{\partial x} = 0$$

If we enter with this formulation into the differential equation, we obtain

$$\left[\frac{\partial\psi_0}{\partial r} + \left(\frac{u_1 - u_0}{u_1}\right)\frac{\partial\psi_1}{\partial r} + \ldots\right]\left[\frac{\partial^2\psi_0}{\partial x\,\partial r} + \left(\frac{u_1 - u_0}{u_1}\right)\frac{\partial^2\psi_1}{\partial x\,\partial r} + \ldots\right]$$

$$-\left[\frac{\partial\psi_0}{\partial x} + \left(\frac{u_1 - u_0}{u_1}\right)\frac{\partial\psi_1}{\partial x} + \ldots\right]\left[\frac{\partial^2\psi_0}{\partial r^2} + \left(\frac{u_1 - u_0}{u_1}\right)\frac{\partial^2\psi_1}{\partial r^2} + \ldots\right]$$

$$+\frac{1}{r}\left[\frac{\partial\psi_0}{\partial x} + \left(\frac{u_1 - u_0}{u_1}\right)\frac{\partial\psi_1}{\partial x} + \ldots\right]\left[\frac{\partial\psi_0}{\partial r} + \left(\frac{u_1 - v_0}{u_1}\right)\frac{\partial\psi_1}{\partial r} + \ldots\right]$$

$$= \epsilon(x)\left\{r\left[\frac{\partial^3\psi_0}{\partial r^3} + \left(\frac{u_1 - u_0}{u_1}\right)\frac{\partial^3\psi_1}{\partial r^3} + \ldots\right]\right.$$

$$\left.-\left[\frac{\partial^2\psi_0}{\partial r^2} + \left(\frac{u_1 - u_0}{u_1}\right)\frac{\partial^2\psi_1}{\partial r^2} + \ldots\right] + \frac{1}{r}\left[\frac{\partial\psi_0}{\partial r} + \left(\frac{u_1 - u_0}{u_1}\right)\frac{\partial\psi_1}{\partial r} + \ldots\right]\right\}$$

If one arranges according to powers of $\left(\frac{u_1 - u_0}{u_1}\right)$ one obtains a series of differential equations for ψ_1, ψ_2, . . .

For ψ_1

$$\frac{\partial\psi_1}{\partial r}\left(\frac{\partial^2\psi_0}{\partial x\,\partial r}\right) + \frac{\partial^2\psi_1}{\partial x\,\partial r}\left(\frac{\partial\psi_0}{\partial r}\right) - \frac{\partial^2\psi_1}{\partial r^2}\left(\frac{\partial\psi_0}{\partial x}\right)$$

$$-\frac{\partial\psi_1}{\partial x}\left(\frac{\partial^2\psi_0}{\partial r^2}\right) + \frac{1}{r}\frac{\partial\psi_1}{\partial r}\left(\frac{\partial\psi_0}{\partial x}\right) + \frac{1}{r}\frac{\partial\psi_1}{\partial x}\left(\frac{\partial\psi_0}{\partial r}\right)$$

$$= \epsilon(x)\left(r\frac{\partial^3\psi_1}{\partial r^3} - \frac{\partial^2\psi_1}{\partial r^2} + \frac{1}{r}\frac{\partial\psi_1}{\partial r}\right)$$

or, taking

$$\frac{\partial\psi_0}{\partial r} = ru_1 \qquad \frac{\partial\psi_0}{\partial x} = 0$$

into consideration

$$\frac{\partial^2\psi_1}{\partial x\,\partial r} r = \epsilon(x)\left(r\ \frac{\partial^3\psi_1}{\partial r^3} - \frac{\partial^2\psi_1}{\partial r^2} + \frac{1}{r}\frac{\partial\psi_1}{\partial r}\right)$$

etc.

<u>On the Theory of Small Disturbances</u>

In the following, we shall limit ourselves at first to small disturbances of the flow field; that is, relatively small differences in velocity $\left(\frac{u_1 - u_0}{u_1}\ \text{small quantity}\right)$.

The velocity field is then defined by the flow potential ψ_1.

Since $\frac{\partial\psi}{\partial r} = r\bar{u}$, the above equation for ψ_1 may be written as follows:

$$r\,\frac{\partial\bar{u}}{\partial x} = \epsilon(x)\left(\frac{\partial\bar{u}}{\partial r} + r\,\frac{\partial^2\bar{u}}{\partial r^2}\right)$$

$$\epsilon(x) = \kappa b(x)\left(\frac{u_1 - u_0}{u_1}\right)$$

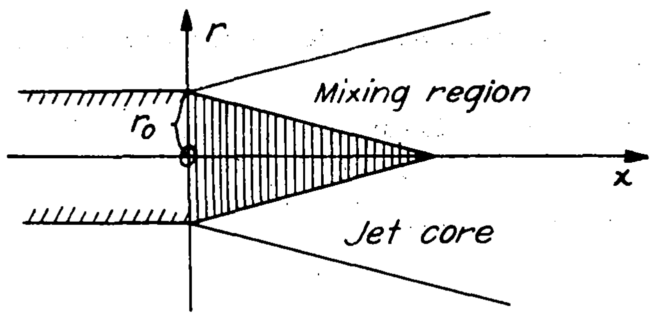
Therewith we have attained for small disturbances a linearization of the equation of motion.

(It should be noted at this point that by the transformation $\mu = \frac{r}{\sqrt{\epsilon(x)}}$ our equation is transformed into

$$\frac{\partial^2\bar{u}}{\partial\mu^2} + \frac{\partial\bar{u}}{\partial\mu}\frac{1}{\mu} - \frac{\partial\bar{u}}{\partial x} = 0$$

With reference to Reichardt's discussions (reference 6), it is of interest to point out that this equation is of the type of a heat conduction equation.)

In view of the conditions existing in our problem



(r_0 = nozzle radius, x = distance from the nozzle in direction of the jet axis), we introduce instead of r the variable $\eta = \frac{r - r_0}{x}$. This coordinate transformation yields

$$\frac{\partial\bar{u}}{\partial r} = \frac{\partial\bar{u}}{\partial\eta}\frac{\partial\eta}{\partial r} = \frac{\partial\bar{u}}{\partial\eta}\frac{1}{x}$$

$$\left(\frac{\partial\bar{u}}{\partial x}\right)_{r=\text{const}} = \left(\frac{\partial\bar{u}}{\partial x}\right)_{\eta=\text{const}} + \frac{\partial\bar{u}}{\partial\eta}\frac{\partial\eta}{\partial x} = \frac{\partial\bar{u}}{\partial x} - \frac{\partial\bar{u}}{\partial\eta}\frac{\eta}{x}$$

thus the equation

$$(x\eta + r_0)\left(\frac{\partial\bar{u}}{\partial x} - \frac{\partial\bar{u}}{\partial\eta}\frac{\eta}{x}\right) = \epsilon(x)\left[\frac{\partial\bar{u}}{\partial\eta}\frac{1}{x} + (x\eta + r_0)\frac{\partial^2\bar{u}}{\partial\eta^2}\frac{1}{x^2}\right]$$

or, respectively, for $\eta + \frac{r_0}{x} \neq 0$

$$\frac{\partial^2 \bar{u}}{\partial \eta^2} + \frac{\partial \bar{u}}{\partial \eta}\left[\frac{1}{\eta + \frac{r_0}{x}} + \eta \frac{x}{\epsilon(x)}\right] - \frac{\partial \bar{u}}{\partial x}\frac{x^2}{\epsilon(x)} = 0$$

$$\epsilon(x) = \kappa b(x)\left(\frac{u_1 - u_0}{u_1}\right)$$

This equation is a linear partial differential equation of the second order of parabolic type.

The solution of this differential equation is fixed unequivocally by the initial condition that for $\frac{x}{r_0} \longrightarrow 0$ the velocity distribution of the plane jet boundary appears.

We first derive (for small disturbances) the velocity distribution of the plane jet rim.

For $\frac{x}{r_0} \longrightarrow 0$ we obtain with the expression $\bar{u}(\eta)$ the equation

$$\frac{d^2\bar{u}}{d\eta^2} + \frac{d\bar{u}}{d\eta}\,\eta\left[\frac{x}{\epsilon(x)}\right]_{\frac{x}{r_0}=0} = 0$$

with

$$\left[\frac{x}{\epsilon(x)}\right]_{\frac{r}{r_0}=0} = \frac{1}{\kappa c\,\frac{u_1 - u_0}{u_1}}$$

Therein $c = \lim\limits_{\frac{x}{r_0}=0} \frac{b(x)}{x}$ and is to be regarded as a function of $\frac{u_1 - u_0}{u_1}$.

With the boundary conditions

$$\bar{u} \longrightarrow \begin{cases} u_1 & \text{for } \eta \longrightarrow -\infty \\ u_0 & \text{for } \eta \longrightarrow +\infty \end{cases}$$

taken into consideration, the integration yields

$$\frac{\bar{u}}{u_1} = \frac{1}{\sqrt{\pi}}\left(\frac{u_0}{u_1} - 1\right)\int_0^{\sigma_0\eta} - e^{-(\sigma_0\eta)^2}\, d(\sigma_0\eta) + \frac{1}{2}\left(1 + \frac{u_0}{u_1}\right)$$

with

$$\sigma_0 = \frac{1}{\sqrt{2\kappa c\left(\frac{u_1 - u_0}{u_1}\right)}}$$

Turning now to our problem, we can expect great difficulties in constructing the exact solution. We limit ourselves therefore to forming an approximate solution. For this purpose we generalize the plane velocity distribution (the initial profile) and set up the following expression

$$\frac{\bar{u}}{u_1} = \frac{1}{\sqrt{\pi}}\left(\frac{u_0}{u_1} - 1\right)\int^{\left[\sigma_1(x)\eta+\sigma_2(x)\right]} e^{-\left[\sigma_1(x)\eta+\sigma_2(x)\right]}\, d\left[\sigma_1(x)\eta + \sigma_2(x)\right] + \frac{1}{2}\left(1 + \frac{u_0}{u_1}\right)$$

This formulation insures at the outset a reasonable shape of the approximation solution.

For σ_1,σ_2 there immediately result, because of the initial condition, the requirements

$$\lim_{\frac{x}{r_0}\to 0} \sigma_1(x) = \sigma_0 \qquad \lim_{\frac{x}{r_0}\to 0} \sigma_2(x) = 0$$

Now the following equation is valid:

$$\sigma_0 = \frac{1}{\sqrt{2\kappa c\left(\frac{u_1 - u_0}{u_1}\right)}} = \lim_{\frac{x}{r_0}\to 0} \frac{1}{\sqrt{2\kappa\left(\frac{u_1 - u_0}{u_1}\right)\frac{b}{x}}}$$

Accordingly, we put

$$\sigma_1(x) = \frac{1}{\sqrt{2\kappa\left(\frac{u_1 - u_0}{u_1}\right)\frac{b}{x}}}$$

$$\left[\frac{\epsilon(x)}{x} = \kappa\left(\frac{u_1 - u_0}{u_1}\right)\frac{b}{x} = \frac{1}{2\sigma_1^2}\right]$$

Furthermore we take care that our approximation statement for small $\frac{x}{r_0}$ yields the exact solution. This will be the case when the $\left(\frac{\partial\bar{u}}{\partial x}\right)_{\frac{x}{r_0}=0}$ of the approximation statement agrees with the $\left(\frac{\partial\bar{u}}{\partial x}\right)$ to be calculated from the differential equation for $\frac{x}{r_0} = 0$.

According to the differential equation:

$$\frac{\partial\bar{u}}{\partial x} = \frac{\epsilon(x)}{x^2}\left\{\frac{\partial^2\bar{u}}{\partial\eta^2} + \frac{\partial\bar{u}}{\partial\eta}\left[\frac{1}{\eta + \frac{r_0}{x}} + \eta\,\frac{x}{\epsilon(x)}\right]\right\}$$

Thus

$$\left(\frac{\partial\bar{u}}{\partial x}\right)_{\frac{x}{r_0}=0} = \lim_{\frac{x}{r_0}\to 0}\frac{\epsilon(x)}{x^2}\left\{\frac{\partial^2\bar{u}}{\partial\eta^2} + \frac{\partial\bar{u}}{\partial\eta}\left[\frac{1}{\eta + \frac{r_0}{x}} + \eta\,\frac{x}{\epsilon(x)}\right]\right\}$$

or, with

$$\frac{\epsilon(x)}{x} = \frac{1}{2\sigma_1^2}$$

$$\left(\frac{\partial\bar{u}}{\partial\frac{x}{r_0}}\right)_{\frac{x}{r_0}=0} = \lim_{\frac{x}{r_0}\to 0}\frac{1}{2\sigma_1^2}\left[\frac{\frac{\partial\bar{u}}{\partial\eta}}{1 + \left(\frac{x}{r_0}\right)\eta} + \frac{\frac{\partial\bar{u}}{\partial\eta^2} + \frac{\partial\bar{u}}{\partial\eta}\eta^2\sigma_1^2}{\left(\frac{x}{r_0}\right)}\right]$$

We now enter into this equation with our approximation expression; that is, we put (except for a common factor)

$$\frac{\partial \bar{u}}{\partial \eta} = e^{-\left[\sigma_1\eta+\sigma_2\right]^2} \sigma_1$$

$$\frac{\partial^2 \bar{u}}{\partial \eta^2} = e^{-\left[\;\right]^2} \left[-2\left(\sigma_1\eta + \sigma_2\right)\sigma_1^2\right]$$

$$\frac{\partial \bar{u}}{\partial \frac{x}{r_0}} = e^{-\left[\;\right]^2} \left(\sigma_1'\eta + \sigma_2'\right)$$

We then consider the relations

$$\lim_{\frac{x}{r_0}=0} \sigma_1 = \sigma_0\left[= \frac{1}{\sqrt{2\kappa c\left(\frac{u_1 - u_0}{u_1}\right)}}\right] \qquad \lim_{\frac{x}{r_0}=0} \sigma_2 = 0$$

furthermore, we assume

$$\lim_{\frac{x}{r_0}\to 0} \sigma_1' = 0$$

The last relation signifies that the width b of the mixing zone is, in the proximity of the nozzle, of rectilinear character, an assumption which seems justified considering the fact that we approach, in the proximity of the nozzle, the conditions of the plane jet boundary.

We then obtain for the left side of the equation

$$\lim_{\frac{x}{r_0}\to 0} \frac{\partial \bar{u}}{\partial \frac{x}{r_0}} = e^{-(\sigma_0\eta)^2} \sigma_2'(0)$$

for the right side

$$\lim_{\frac{x}{r_0}\to 0} \frac{1}{2\sigma_1^2}\left[\frac{\frac{\partial \bar{u}}{\partial \eta}}{1+\left(\frac{x}{r_0}\right)\eta} + \frac{\frac{\partial^2 \bar{u}}{\partial \eta^2} + \frac{\partial \bar{u}}{\partial \eta}\eta^2\sigma_1^2}{\left(\frac{x}{r_0}\right)}\right] = \frac{1}{2\sigma_0^3}\left[\sigma_0 - 2\sigma_0^2\sigma_2'(0)\right]e^{-(\sigma_0\eta)^2}$$

Equating yields the equation

$$\sigma_2'(0) = \frac{1}{2\sigma_0^2}\left[\sigma_0 - 2\sigma_0^2\sigma_2'(0)\right]$$

or respectively,

$$\sigma_2'(0) = \frac{1}{4\sigma_0}$$

This results in $\sigma_2 = \frac{1}{4\sigma_0}\left(\frac{x}{r_0}\right)$ for small $\frac{x}{r_0}$.

This guarantees first of all that our approximation expression for $\frac{x}{r_0} \longrightarrow 0$ represents the exact solution.

If we enter with the approximation expression thus constructed into the differential equation, we recognize immediately that the latter (due to the factor $e^{-\left[\sigma_1\eta+\sigma_2\right]^2}$ is satisfied also for $\eta \longrightarrow \infty$ (and arbitrary $\frac{x}{r_0}$).

Thus our approximate expression with σ_1, σ_2 fixed in the above manner yields a function which corresponds in boundary zones of the region to the exact solution.

As to the behavior of our function in the interior of the region, it is found that the function in case of suitable "continuation" into the interior of the region satisfies the differential equation along $\eta = 0$.

For $\underline{\eta = 0}$ the differential equation reads

$$\frac{\partial \bar{u}}{\partial \frac{x}{r_0}} = \frac{\epsilon(x)}{x}\left(\frac{r_0}{x}\right)\left[\frac{\partial^2 \bar{u}}{\partial \eta^2} + \frac{\partial \bar{u}}{\partial \eta}\left(\frac{x}{r_0}\right)\right]$$

If one enters with the approximation expression and considers

$$\frac{\epsilon(x)}{x} = \frac{1}{2\sigma_1^{\,2}}$$

one obtains

$$\sigma_2' = \frac{1}{2\sigma_1^{\,2}}\left(\frac{r_0}{x}\right)\left[-2\sigma_2\sigma_1^{\,2} + \sigma_1\left(\frac{x}{r_0}\right)\right]$$

or

$$\sigma_2' + \sigma_2 \frac{1}{\left(\frac{x}{r_0}\right)} = \frac{1}{2\sigma_1}$$

As solution one obtains

$$\sigma_2 = \frac{1}{2}\frac{1}{\left(\frac{x}{r_0}\right)}\int_0^{\frac{x}{r_0}}\left(\frac{x}{r_0}\right)\frac{1}{\sigma_1}\, d\left(\frac{x}{r_0}\right)$$

For small $\frac{x}{r_0}$ one has again

$$\sigma_2 = \frac{1}{4}\frac{1}{\sigma_0}\left(\frac{x}{r_0}\right)$$

We may also write

$$\sigma_2 = \frac{1}{2}\sqrt{2\kappa\left(\frac{u_1 - u_0}{u_1}\right)}\frac{1}{\left(\frac{x}{r_0}\right)}\int_0^{\frac{x}{r_0}}\left(\frac{x}{r_0}\right)\sqrt{\frac{b}{x}}\; d\left(\frac{x}{r_0}\right)$$

Therewith we have obtained for small disturbances the following approximation function

$$\frac{\bar{u}}{u_1} = \frac{1}{\sqrt{\pi}}\left(\frac{u_0}{u_1} - 1\right)\int_0^{\left[\sigma_1\eta+\sigma_2\right]} e^{-\left[\sigma_1\eta+\sigma_2\right]^2}\, d\left(\sigma_1\eta + \sigma_2\right) + \frac{1}{2}\left(1 + \frac{u_0}{u_1}\right)$$

where

$$\sigma_1(x) = \frac{1}{\sqrt{2\kappa\left(\frac{u_1 - u_0}{u_1}\right)\frac{b}{x}}}$$

$$\sigma_2(x) = \frac{1}{2}\sqrt{2\kappa\left(\frac{u_1 - u_0}{u_1}\right)}\frac{1}{\left(\frac{x}{r_0}\right)}\int_0^{\frac{x}{r_0}}\left(\frac{x}{r_0}\right)\sqrt{\frac{b}{x}}\,d\left(\frac{x}{r_0}\right)$$

To sum up: This function satisfies the differential equation with the initial conditions prescribed for small $\frac{x}{r_0}$ as well as for large positive η; in the interior of the region it satisfies the differential equation along the jet $\eta = 0$. Therewith we have constructed an approximate function which in boundary zones of the region of integration and in its interior along the jet $\eta = 0$ is to be regarded as exact solution.

On the Theory of Larger Disturbances

Let us now consider larger disturbances $\frac{u_1 - u_0}{u_1}$ not a small quantity .

First, we shall treat the problem of the initial profile.

Görtler's calculations (reference 3) showed that even the first approximation $\left(\text{for small } \frac{u_1 - u_0}{u_1}\right)$ represents, purely with respect to shape, a very good approximation. This applies, however, only to the shape of the distribution curve - not to its position. The velocity distribution calculated by Görtler is unequivocally fixed by the arbitrary requirement that $\bar{u}(0) = \frac{u_1 - u_0}{2}$. However, Görtler points out that with $\bar{u}(\eta)$, $\bar{v}(\eta)$, the equations $u^* = \bar{u}(\eta + a)$, $v^* = \bar{v}(\eta + a) - a\bar{u}(\eta + a)$ also represents a system of solution. But this remaining uncertainty is here eliminated by the fact that for the jet core the transverse component $\bar{v}$ must vanish as follows from the continuity.

If $\bar{u}(\eta)$, $\bar{v}(\eta)$ is the velocity distribution calculated by Görtler which is characterized by $\bar{u}(0) = \frac{u_1 - u_0}{2}$, the quantity a must therefore be determined in such a manner that $v_1 - au_1 = 0$ which yields $a = \frac{v_1}{u_1}$. Taking Görtler's calculations as a basis, one obtains in first approximation

$$\frac{v_1}{u_1} = -\frac{1}{\sigma}\left[\left(\frac{u_1 - u_0}{u_1}\right)0.36\right]$$

thus

$$\sigma a = -0.36\left(\frac{u_1 - u_0}{u_1}\right)$$

Therewith the initial profile for all $\frac{u_1 - u_0}{u_1}$ is unequivocally determined. If we base the shape representation on the first approximation, the initial profile is

$$\frac{\bar{u}}{u_1} = \frac{1}{\sqrt{\pi}}\left(\frac{u_0}{u_1} - 1\right)\int_0^{\xi} e^{-\xi^2}\, d\xi + \frac{1}{2}\left(1 + \frac{u_0}{u_1}\right)$$

where

$$\xi = \sigma_0\eta - 0.36\,\frac{u_1 - u_0}{u_1}$$

and

$$\sigma_0 = \frac{1}{\sqrt{2\kappa c\left(\frac{u_1 - u_0}{u_1}\right)}}$$

For the further development of the profiles in the core region, starting from this initial profile, we take as a basis the regularity found for small disturbances.

$$\frac{\bar{u}}{u_1} = \frac{1}{\sqrt{\pi}}\left(\frac{u_0}{u_1} - 1\right)\int_0^{\eta^*} e^{-\eta^{*2}}\, d\eta^* + \frac{1}{2}\left(1 + \frac{u_0}{u_1}\right)$$

where

$$\eta^* = \sigma_1\eta - 0.36\,\frac{u_1 - u_0}{u_1} + \sigma_2$$

with the terms $\sigma_1(x)$, $\sigma_2(x)$ determined before.

This function therefore yields the initial profile in first approximation. How far it may be regarded as approximation in the region is not investigated in more detail here. For $\frac{u_1 - u_0}{u_1} \longrightarrow 0$ it is transformed into the approximate function found for small disturbances.

Our approximate function generalized to arbitrary disturbances therefore reads

$$\frac{\bar{u}}{u_1} = \frac{1}{\sqrt{\pi}}\left(\frac{u_0}{u_1} - 1\right)\int_0^{\eta^*} e^{-\eta^{*2}}\,d\eta^* + \frac{1}{2}\left(1 + \frac{u_0}{u_1}\right)$$

where

$$\eta^* = \sigma_1\eta - 0.36\,\frac{u_1 - u_0}{u_1} + \sigma_2$$

with

$$\sigma_1(x) = \frac{1}{\sqrt{2\kappa\left(\frac{u_1 - u_0}{u_1}\right)\frac{b}{x}}}$$

$$\sigma_2(x) = \frac{1}{2}\sqrt{2\kappa\left(\frac{u_1 - u_0}{u_1}\right)}\,\frac{1}{\left(\frac{x}{r_0}\right)}\int_0^{\frac{x}{r_0}}\left(\frac{x}{r_0}\right)\sqrt{\frac{b}{x}}\,d\left(\frac{x}{r_0}\right)$$

The coordination to η is obtained by

$$\eta = \frac{\eta^* + 0.36\,\frac{u_1 - u_0}{u_1}}{\sigma_1} - \frac{\sigma_2}{\sigma_1}$$

where

$$\frac{\sigma_2}{\sigma_1} = \frac{1}{2}\,\frac{1}{\sigma_1{}^2}\,\frac{\displaystyle\int_0^{\frac{x}{r_0}}\left(\frac{x}{r_0}\right)\sqrt{\frac{b}{x}}\,d\left(\frac{x}{r_0}\right)}{\left(\frac{x}{r_0}\right)\sqrt{\frac{b}{x}}}$$

Thus the curves result from one another by similarity transformations.

Calculation of the curves requires, furthermore, knowledge of the functions $\sigma_1(x)$, $\sigma_2(x)$ and, respectively, of the mixing width $b(x)$ and the constant κ. These quantities result from the approximate calculation (carried out with the aid of the momentum theorem) for the dimensions of the core region.

Figure 1 contains for the parameter values

$$\frac{u_1 - u_0}{u_1} = 1.0;\ 0.8;\ 0.6;\ 0.4;\ 0.2$$

the velocity distributions $\frac{\bar{u}}{u_1}$ calculated for $\frac{x}{r_0} = 0$ and the core end.

In figure 2 the functions $\sigma_1(x)$ and $\sigma_2(x)$ are plotted for the parameter values named above, as functions of $\frac{x}{r_0}$ up to the core end.

Calculation of the Transverse Component

The transverse component $\bar{v}$ of the flow is determined from the continuity equation

$$\frac{\partial(r\bar{u})}{\partial x} + \frac{\partial(r\bar{v})}{\partial r} = 0$$

and, respectively

$$\bar{v} = -\frac{1}{r}\int\left(r\,\frac{\partial\bar{u}}{\partial x}\right)dr \qquad r \neq 0$$

Transformation of r into $\eta = \frac{r - r_0}{x}$ results in

$$\overline{v} = - \frac{1}{\left(\eta + \frac{r_0}{x}\right)} \int^{\eta} \left(\eta + \frac{r_0}{x}\right)\left(x \frac{\partial \overline{u}}{\partial x} - \eta \frac{\partial \overline{u}}{\partial \eta}\right) d\eta \qquad \left(\eta + \frac{r_0}{x} \neq 0\right)$$

The integration constant is determined from the requirement that in the jet core the transverse component $\overline{v}$ must vanish.

As the lower limit we choose accordingly the η_k determined by the bounding of the jet core (concerning the dimensions of the core region, compare next paragraph).

In order to avoid complicating the calculation, a rectilinear course of the mixing width $b(x)$ was assumed. This assumption is approximately correct. (Compare fig. 11.)

For the velocity distribution $\frac{\overline{u}}{u_1}$ we substitute our approximate function. The performance of the calculation (appendix no. 1) yields the following final formula.

$$\frac{\overline{v}}{u_1} = - \frac{1}{\left(\eta + \frac{r_0}{x}\right)} \frac{1}{\sqrt{\pi}}\left(\frac{u_0}{u_1} - 1\right)\left[\frac{1}{2} \frac{1}{\sigma_1{}^2}\left(\frac{x}{r_0}\right)\left\{I\right\} + \frac{\sqrt{\pi}}{2} \frac{1}{\sigma_1}\left\{II\right\}\right] +$$

$$\frac{1}{\left(\eta + \frac{r_0}{x}\right)} \frac{1}{\sqrt{\pi}}\left(\frac{u_0}{u_1} - 1\right)\left[\frac{1}{2} \frac{1}{\sigma_1{}^2}\left(\frac{r_0}{x}\right)\left\{I\right\} + \frac{1}{2} \frac{1}{\sigma_1}\left\{III\right\}\right]$$

where

$$\left\{I\right\} = \left\{- e^{-[\,]^2} + e^{-[\,]_0{}^2} - \left(\sigma_2 - 0.36 \frac{u_1 - u_0}{u_1}\right)\sqrt{\pi}\left(F_1{}'_{[\,]} + F_1{}'_{-[\,]}\right)\right\}$$

$$\left\{II\right\} = \left\{F_1{}'_{[\,]} + F_1{}'_{-[\,]_0}\right\}$$

$$\left\{III\right\} = \left\{-[\,]\ e^{-[\,]^2} + [\,]_0\ e^{-[\,]_0^2} + 2\left(\sigma_2 - 0.36\,\frac{u_1 - u_0}{u_1}\right)\left(e^{-[\,]^2} + e^{-[\,]_0^2}\right) + \right.$$

$$\left. \sqrt{\pi}\left[\frac{1}{2} + \left(\sigma_2 - 0.36\,\frac{u_1 - u_0}{u_1}\right)^2\right]\left(F_1{}'_{[\,]} + F_1{}'_{-[\,]_0}\right)\right\}$$

Therein

$$[\,] = \left(\sigma_1\eta - 0.36\,\frac{u_1 - u_0}{u_1} + \sigma_2\right)$$

$$[\,]_0 = \left(\sigma_1{}'\eta_k - 0.36\,\frac{u_1 - u_0}{u_1} + \sigma_2\right)$$

$$F_1{}' = \frac{2}{\sqrt{\pi}}\int_0^{[\,]} e^{-[\,]^2}\, d[\,]$$

and $F_1{}'_{[\,]}$ and $F_1{}'_{-[\,]_0}$, respectively, signify the values of the error integral taken at the points $[\,]$ and $-[\,]_0$, respectively.

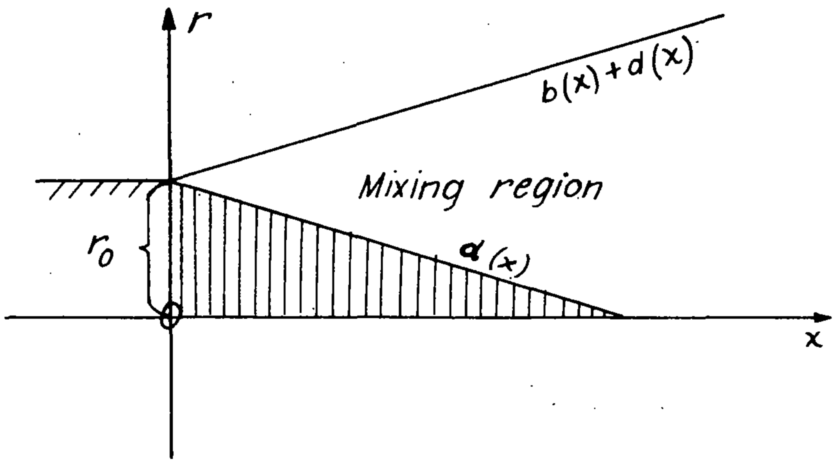
In figures 3 to 7 the distributions of the transverse component for a section $\left(\frac{x}{r_0}\right) = 0.1$ near the nozzle and a section of 3/4 of the core length are plotted for the parameter values $\frac{u_1 - u_0}{u_1} = 1.0, 0.8, 0.6, 0.4, 0.2$.

In the case $\frac{u_1 - u_0}{u_1} = 1.0$ there are shown, moreover, the distributions for the sections 1/4 of the core length and the core end itself.

(Remark: The transverse components calculated for the core end seem to yield too small values of the approach flow; the reason is that the poor approximation of the velocity distribution, an essential characteristic of the Prandtl expression, in the boundary zones takes the more effect in the calculation of the $\overline{v}$ component the more one approaches the core end.)

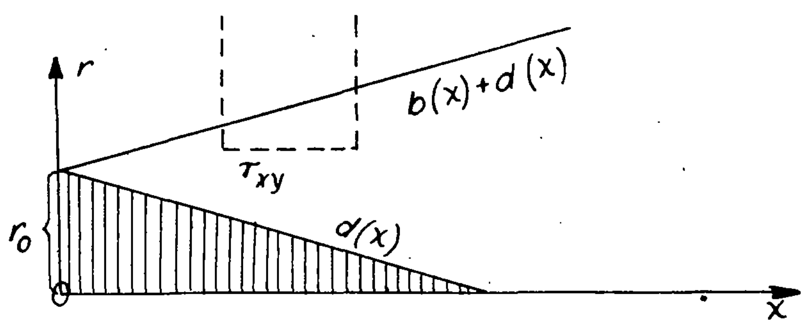
For small $\frac{u_1 - u_0}{u_1}$, the transverse component becomes very small (note the different scales in the various representations).

(b) The Dimensions of the Core Region



The dimensions of the core region are defined by the limiting curve of the jet core $d(x)$ and the width of the mixing zone $b(x)$ or, respectively, the outer limiting curve of the latter $b(x) + d(x)$.

According to Kuethe's procedure (reference 1) we take as a basis the theorem of momentum in Tollmien's formulation (reference 4).



If one marks off a control area in the indicated manner, one obtains in the rotationally symmetrical case

$$\left(\overline{u} - u_0\right) \overline{v}r - \frac{\partial}{\partial x}\left[\int_r^{\infty} \overline{u}\left(\overline{u} - u_0\right) r\, dr\right] = \left(r\tau_{xy}\right)$$

u_0 = velocity of the medium surrounding the jet.

According to the more recent Prandtl expression

$$\tau_{xy} = \kappa b(x)\left(u_1 - u_0\right) \frac{\partial \overline{u}}{\partial r}$$

Thus we obtain, if we, furthermore, take the limits of the mixing zone into consideration

$$\left(\overline{u} - u_0\right) \overline{v}r - \frac{\partial}{\partial x}\left[\int_r^{d+b} \overline{u}\left(\overline{u} - u_0\right) r\, dr\right] = r\kappa b\left(u_1 - u_0\right) \frac{\partial \overline{u}}{\partial r}$$

According to the existing conditions we transform (according to Kuethe) with

$$\eta = \frac{r - d(x)}{b(x)}$$

Then

$$\eta = \frac{r - d(x)}{b(x)}$$

$$r = b\eta + d$$

$$\frac{\partial \eta}{\partial x} = -\frac{d'}{b} - \eta \frac{b'}{b}$$

If we make, in addition, the assumption that $\overline{u}$ depends only on η, not on x, there follows

$$\frac{\partial}{\partial x}\int_r^{d+b} = \int_\eta^1 \left\{\frac{d}{d\eta}\left[\overline{u}\left(\overline{u} - u_0\right)\right]\right\}\left[-\eta^2 bb' - \eta(bd' + b'd) - dd'\right] d\eta$$

For $\bar{v}$ we finally insert the continuity equation

$$\bar{v} = -\frac{1}{r}\int_d^r \left(r\frac{\partial\bar{u}}{\partial x}\right)dr$$

or

$$\bar{v} = -\frac{1}{(\eta b + d)}\int_0^\eta \frac{\partial\bar{u}}{\partial\eta}\left[-\eta^2 bb' - \eta(bd' + b'd) - dd'\right]d\eta$$

For approximate calculation, we write for the velocity distribution the sample expression

$$\frac{\bar{u} - u_0}{u_1} = \left(\frac{u_1 - u_0}{u_1}\right)(1 - \eta)$$

$$\eta = \frac{r - d(x)}{b(x)}$$

This expression, which may be regarded as a first rough approximation for the velocity distribution, will probably lead to not too large errors for the integral calculation. The value $\frac{\partial\bar{u}}{\partial r}$ determining the shearing stress also will probably result in a usable approximation for the central region of the mixing zone.

The result is

$$\frac{\partial\frac{\bar{u}}{u_1}}{\partial r} = -\frac{1}{b}\,\frac{u_1 - u_0}{u_1}$$

$$\frac{d}{d\eta}\left[\frac{\bar{u}}{u_1}\left(\frac{\bar{u} - u_0}{u_1}\right)\right] = -2\left(\frac{u_1 - u_0}{u_1}\right)^2(1 - \eta) - \frac{u_0}{u_1}\left(\frac{u_1 - u_0}{u_1}\right)$$

(1) We now put $r = 0$. The momentum theorem is then transformed by integration into the form of the theorem of conservation of momentum

$$\int_0^{b+d} \bar{u}\left(\bar{u} - u_0\right)r\,dr = \text{const}$$

or

$$\int_0^{d(x)} u_1\left(u_1 - u_0\right)r\,dr + \int_d^{d+b} \bar{u}\left(\bar{u} - u_0\right)r\,dr = u_1\left(u_1 - u_0\right)\frac{r_0^{\,2}}{2}$$

$$u_1\left(u_1 - u_0\right)\frac{d^2}{2} + \int_0^1 \bar{u}\left(\bar{u} - u_0\right)(\eta b + d)b\,d\eta = u_1\left(u_1 - u_0\right)\frac{r_0^{\,2}}{2}$$

$$\frac{1}{2}\left(\frac{u_1 - u_0}{u_1}\right)\left(d^2 - r_0^{\,2}\right) + b^2\int_0^1 \frac{\bar{u}}{u_1}\left(\frac{\bar{u} - u_0}{u_1}\right)\eta\,d\eta + bd\int_0^1 \frac{\bar{u}}{u_1}\left(\frac{\bar{u} - u_0}{u_1}\right)d\eta = 0$$

If one inserts $\dfrac{\bar{u}}{u_1} = \left(\dfrac{u_1 - u_0}{u_1}\right)(1 - \eta)$ and carries out the integration, one obtains

$$\frac{1}{2}\left(d^2 - r_0^{\,2}\right) + b^2\left[\left(\frac{u_1 - u_0}{u_1}\right)\frac{1}{12} + \frac{u_0}{u_1}\,\frac{1}{6}\right] + bd\left(\frac{u_1 - u_0}{u_1}\,\frac{1}{3} + \frac{u_0}{u_1}\,\frac{1}{2}\right) = 0$$

or

$$\frac{b^2}{3}\left[1 - \frac{1}{2}\left(\frac{u_1 - u_0}{u_1}\right)\right] + bd\left[1 - \frac{1}{3}\left(\frac{u_1 - u_0}{u_1}\right)\right] + d^2 = r_0^{\,2}$$

(2) In order to obtain a second equation between b and d, we put $r = r_0$.

If one performs the somewhat lengthy elementary calculation, one obtains finally (compare appendix no. 2)

$$\begin{aligned} &bb'\left\{\frac{u_1 - u_0}{u_1}\left[-\frac{1}{6}\left(\frac{r_0 - d}{b}\right)^4 + \frac{1}{6}\right] + \frac{1}{3}\left(\frac{r_0 - d}{b}\right)^3 - \frac{1}{3}\right\} + \\ &\quad (b'd + bd')\left\{\left(\frac{u_1 - u_0}{u_1}\right)\left[-\frac{1}{6}\left(\frac{r_0 - d}{b}\right)^3 + \frac{1}{6}\right] + \frac{1}{2}\left(\frac{r_0 - d}{b}\right)^2 - \frac{1}{2}\right\} + \\ &\quad dd'\left[\left(\frac{r_0 - d}{b}\right) - 1\right] = -\,r_0\kappa\left(\frac{u_1 - u_0}{u_1}\right) \end{aligned}$$

The theorem of conservation of momentum reads in differentiated form. (Compare (1).)

$$bb'\left[\frac{1}{3} - \frac{1}{6}\left(\frac{u_1 - u_0}{u_1}\right)\right] + (b'd + bd')\left[\frac{1}{2} - \frac{1}{6}\left(\frac{u_1 - u_0}{u_1}\right)\right] + dd'\left\{1\right\} = 0$$

By addition of the two equations one obtains

$$bb'\left[\frac{1}{3}\left(\frac{r_0 - d}{b}\right)^3 - \frac{1}{6}\left(\frac{u_1 - u_0}{u_1}\right)\left(\frac{r_0 - d}{b}\right)^4\right] + (b'd + bd')\left[\frac{1}{2}\left(\frac{r_0 - d}{b}\right)^2 - \frac{1}{6}\left(\frac{u_1 - u_0}{u_1}\right)\left(\frac{r_0 - d}{b}\right)^3\right] + dd'\left\{\frac{r_0 - d}{b}\right\} = - r_0\kappa\left(\frac{u_1 - u_0}{u_1}\right)$$

(3) We now proceed to determine b and d from the two equations obtained. We replace b in the second equation by the expression for the function which we obtain by solving the first equation with respect to b.

$$\frac{b}{r_0} = a_0\left(\frac{d}{r_0}\right) + \sqrt{a_1 - a_2\left(\frac{d}{r_0}\right)^2}$$

where

$$a_0 = -\frac{3}{2}\,\frac{1 - \frac{1}{3}\left(\frac{u_1 - u_0}{u_1}\right)}{1 - \frac{1}{2}\left(\frac{u_1 - u_0}{u_1}\right)}$$

$$a_1 = \frac{3}{1 - \frac{1}{2}\left(\frac{u_1 - u_0}{u_1}\right)}$$

$$a_2 = \frac{3}{1 - \frac{1}{2}\left(\frac{u_1 - u_0}{u_1}\right)} - \frac{9}{4}\left[\frac{1 - \frac{1}{3}\left(\frac{u_1 - u_0}{u_1}\right)}{1 - \frac{1}{2}\,\frac{u_1 - u_0}{u_1}}\right]^2$$

$$b' = \left(a_0 - \frac{a_2 d}{\sqrt{a_1 - a_2 d^2}}\right) d'$$

Substitution then yields

$$d' = \left[f_1 - \left(\frac{u_1 - u_0}{u_1}\right) f_2\right] = -\kappa\left(\frac{u_1 - u_0}{u_1}\right)$$

with

$$f_1 = \frac{\left[1 - \left(\frac{d}{r_0}\right)\right]^2 \left[\frac{1}{3} + \frac{1}{6}\left(\frac{d}{r_0}\right)\right]}{\left[a_0\left(\frac{d}{r_0}\right) + \sqrt{a_1 - a_2\left(\frac{d}{r_0}\right)^2}\right]^2} \left[a_0 - \frac{a_2\left(\frac{d}{r_0}\right)}{\sqrt{}}\right] + \frac{1}{2} \frac{1 - \left(\frac{d}{r_0}\right)^2}{\left[a_0\left(\frac{d}{r_0}\right) + \sqrt{}\right]}$$

$$f_2 = \frac{\frac{1}{6}\left[1 - \left(\frac{d}{r_0}\right)\right]^3 \left[a_0 - \frac{a_2\left(\frac{d}{r_0}\right)}{\sqrt{}}\right]}{\left[a_0\left(\frac{d}{r_0}\right) + \sqrt{}\right]^3} + \frac{1}{6} \frac{\left[1 - \left(\frac{d}{r_0}\right)\right]^3}{\left(a_0 \frac{d}{r_0} + \sqrt{}\right)^2}$$

We obtain $\frac{x}{r_0}$ as a function of $\frac{d}{r_0}$

$$d(x) = -\frac{1}{\kappa} \frac{1}{\left(\frac{u_1 - u_0}{u_1}\right)} \left(f_1 - \frac{u_1 - u_0}{u_1} f_2\right) d(d)$$

$$\left(\frac{x}{r_0}\right) = -\frac{1}{\kappa} \frac{1}{\left(\frac{u_1 - u_0}{u_1}\right)} \int_1^{\frac{d}{r_0}} \left(f_1 - \frac{u_1 - u_0}{u_1} f_2\right) d\left(\frac{d}{r_0}\right)$$

The evaluation of the integral could, in itself, be carried out by analytical method since the integrand is built rationally in $\frac{d}{r_0}$ and a square root. However, the breaking up into partial fraction which has to be done in this procedure is very troublesome. Hence it is advisable to perform the evaluation graphically.

For $\frac{d}{r_0} = 1$ the integrand $\left[f_1 - \left(\frac{u_1 - u_0}{u_1}\right)f_2\right]$ assumes the indefinite expression $\frac{0}{0}$. The limiting value is

$$\lim_{\frac{d}{r_0}\to 1}\left(f_1 - \frac{u_1 - u_0}{u_1} f_2\right) = \frac{1}{2}\,\frac{\left(a_0 + \frac{a_2}{a_0}\right)}{\left(\frac{a_1}{a_0}\right)^2} + \frac{1}{6}\left(\frac{u_1 - u_0}{u_1}\right)\frac{\left(a_0 + \frac{a_2}{a_0}\right)}{\left(\frac{a_1}{a_0}\right)^3} - \frac{1}{\left(\frac{a_1}{a_0}\right)}$$

If $\frac{d}{r_0}$ was determined, analytically or graphically, as a function of $\frac{x}{r_0}$, $b(x)$ results from

$$\frac{b}{r_0} = a_0\,\frac{d}{r_0} + \sqrt{a_1 - a_2\left(\frac{d}{r_0}\right)^2}$$

The relation

$$\left(\frac{db}{dx}\right)_{\frac{x}{r_0}=0} = -\kappa\left(\frac{u_1 - u_0}{u_1}\right)\left(a_0 + \frac{a_2}{a_0}\right)\frac{1}{\lim\limits_{\frac{d}{r_0}\to 1}\left[f_1 - \left(\frac{u_1 - u_0}{u_1}\right)f_2\right]}$$

(which by comparison with measurements on the plane jet boundary may serve for the determination of κ) also is of interest.

The symbol κ appears as the only empirical constant.

With the measured results on the plane jet boundary with zero outer velocity (given by Tollmien (reference 4)) as a basis, there results with

$$\left(\frac{db}{dx}\right)_{\frac{x}{r_0}=0} = 0.255$$

$$0.255 = -\kappa(-3)\frac{1}{0.1854}$$

$$\kappa = \underline{\underline{0.01576}}$$

Examples:

In figure 8 the dimensions of the corresponding core region are represented for the parameter values

$$\frac{u_1 - u_0}{u_1} = 1.0,\ 0.8,\ 0.6,\ 0.4,\ 0.2$$

Figure 9 contains the core lengths $\frac{x_k}{r_0}$, figure 10 the mixing widths $\frac{b_k}{r_0}$ at the core end as functions of $\frac{u_1 - u_0}{u_1}$.

Figure 11 shows the mixing widths $\frac{b}{r_0}$ for the various parameter values of $\frac{u_1 - u_0}{u_1}$ as functions of $\frac{x}{r_0}$.

Figure 12 represents the angle of spread of the respective mixing region $c = \left(\frac{db}{dx}\right)_{\frac{x}{r_0}=0}$.

Figure 13 represents $\sigma_0 = \frac{1}{\sqrt{2\kappa c \frac{u_1 - u_0}{u_1}}}$ as a function of $\frac{u_1 - u_0}{u_1}$, with Tollmien's value $c = 0.255$ for $\frac{u_1 - u_0}{u_1} = 1$ being the defining quantity.

Figure 14 shows for the medium at rest $\left(\frac{u_1 - u_0}{u_1} = 1\right)$ the quantity κ as a function of $c = \left(\frac{db}{dx}\right)_{\frac{x}{r_0}=0}$. Figure 15 shows $\sigma_0 = \frac{1}{\sqrt{2\kappa c}}$ as a function of $c = \left(\frac{db}{dx}\right)_{\frac{x}{r_0}=0}$.

Figure 16, finally, contains the limiting value

$$\lim_{\frac{d}{r_0} \to 1} \left[f_1 - \left(\frac{u_1 - u_0}{u_1}\right) f_2\right]$$

necessary for calculation of the integrand in

$$\frac{x}{r_0} = -\frac{1}{\kappa \cdot \left(\frac{u_1 - u_0}{u_1}\right)} \int_1^{\frac{d}{r_0}} \left[f_1 - \left(\frac{u_1 - u_0}{u_1}\right) f_2\right] d\left(\frac{d}{r_0}\right)$$

III. COMPARISON WITH MEASUREMENTS

Measurements on a free jet issuing from a nozzle and spreading in moving air of the same temperature do not exist so far.

In order to test the theory by experiment, a measurement for the case $\frac{u_1 - u_0}{u_1} = 0.5$ was performed at the Focke-Wulf plant.

The measurements were carried out with the test arrangement with the 5 millimeter nozzle described in reference 5. A certain experimental difficulty was experienced in producing temperature equality in the two jets; it was achieved by regulation of the combustion chamber temperature with the test chamber pressure p_k and the probe pressure p_s kept constant. However a perfect agreement of the jet temperatures could not be accomplished inasmuch as the temperature measurement performed with a thermoelement is rather inaccurate in this low region.

The test data were:

Outer jet:

Static pressure (Measured relative to atmospheric pressure)	p_k = -100 mm Hg
Room temperature	$t_0 = 20^{\circ}$
Barometer reading	p_0 = 754.5 mm Hg

Inner jet:

Total pressure $p_s = 340$ mm Hg
(Measured relative to atmospheric pressure

Stagnation temperature $t_s = 59^\circ$

The evaluation of the measured values was made according to the adiabatic

$$T_1 = T_2\left(\frac{p_1}{p_2}\right)^{\frac{\kappa-1}{\kappa}}$$

and the efflux equation

$$u_1 = \sqrt{\frac{2\kappa}{\kappa - 1}\, gRT_2\left[1 - \left(\frac{p_1}{p_2}\right)^{\frac{\kappa-1}{\kappa}}\right]}$$

with constant static pressure assured in the mixing region.

Due to the imperfect readability of the thermoelement which, as mentioned before, is too rough for smaller temperature differences, it was impossible to measure the distribution of the stagnation temperatures over the mixing region. For the evaluation a linear drop of the stagnation temperatures along the mixing width was assumed.

For the outer jet there results

$t_A = 9^\circ$ $\bar{u}_A = 151$ meters per second

for the jet issuing from the inner nozzle

$t_i = 13^\circ$ $\bar{u}_i = 302$ meters per second

The inner jet therefore has, compared to the outer jet, an excess temperature of 4°. For the velocity ratio the result $\dfrac{u_1 - u_0}{u_1} = 0.5$ was obtained.

Figure 17 shows the total pressure distribution $\dfrac{\dfrac{p_s}{p_B} + \dfrac{p_k}{p_B}}{\left(\dfrac{p_s}{p_B} + \dfrac{p_k}{p_B}\right)_{central}}$, made dimensionless with the central value, for the various test sections. The section near the nozzle which still shows the character of a turbulent pipe flow is represented in figure 18. Figure 19 shows, in addition, the variation of the total pressures along the jet axis.

Figures 20 to 22 contain the corresponding representations for the velocities made dimensionless by the velocity u_1 of the jet issuing from the nozzle.

As to the comparison with the theory, it must be noted that the velocity distribution at the exit from the nozzle is not rectangular, as assumed in the theory, but that it represents the profile of a turbulent pipe flow. (Compare fig. 21.) Hence it proves necessary to introduce the conception of the "effective diameter" in contrast to the geometric diameter.

We define the effective nozzle diameter as the width of the rectangular velocity distribution of the amount u_1 which is equivalent to the existing momentum distribution. That is, we calculate the effective nozzle diameter from the equation

$$u_1\left(u_1 - u_0\right) \frac{r^2_{effect.}}{2} = \int_0^\infty \bar{u}\left(\bar{u} - u_0\right) r \, dr$$

with the integral, which according to the theorem of conservation of momentum represents a constant, to be extended over an arbitrary cross section.

In our case the integration over the cross section near the nozzle yields

$$r_{effect.} = 0.945 r_{geom.}$$

Whereas the plotting over $\eta = \frac{r - r_{geom.}}{x}$ lets the test points appear as still lying on one curve, the plotting over $\eta = \frac{r - r_{effect.}}{x}$ results in a stagger of the velocity distributions with increasing $\frac{x}{r_0}$ toward negative η. This stagger toward negative η expresses the immediately obvious fact that the isotacs of the flow field are curved toward negative η (toward the jet axis).

Figure 23 contains the theoretical curves for $\frac{x}{r_0} = 0$ and $\frac{x_k}{r_0}$ (the core end); in addition, the test points of the sections $x = 10$ millimeters and $x = 45$ millimeters were plotted. The agreement appears to be good as far as the velocity gradient and the orientation in space in the central mixing region are concerned; the agreement in the transitions toward the jet core and the surrounding medium is less satisfactory. Deviations in these transitions are essential characteristics of the more recent Prandtl expression, but are caused here probably mainly by the approximation character of our developments.

For the core length there results according to the theory a value of $x_k = 22.0 r_{effect}$, whereas the measurements along the jet axis (compare fig. 22) result in about $x_k = \frac{19}{0.945} = 20.1 r_{effect}$. It has to be noted that the experimental determination of the core end is affected by some uncertainty.

IV. SUMMARY

The spreading of a free rotationally symmetrical jet issuing from a nozzle represents a turbulent flow state.

The theoretical investigation is based on the more recent Prandtl expression $\epsilon = \kappa b \left|\bar{u}_{max} - \bar{u}_{min}\right|$ for the momentum transport. The continuity equation and the equation of momentum are at disposal for calculation of the velocity distribution. In case of limitation to small disturbances $\left(\frac{u_1 - u_0}{u_1}\right.$ small quantity, where u_1 is jet exit velocity, u_0 velocity of the surrounding medium$\Big)$ the equation of momentum may be linearized

$$r \frac{\partial \bar{u}}{\partial x} = \epsilon(x)\left(\frac{\partial \bar{u}}{\partial r} + r \frac{\partial^2 \bar{u}}{\partial r^2}\right)$$

An approximate solution is constructed which is characterized by the fact that in boundary zones of the region as well as along the jet $\eta = 0$ in the interior of the region it has to be regarded as exact solution.

For arbitrary disturbances $\left(\frac{u_1 - u_0}{u_1} \text{ arbitrary} > 0\right)$ the initial profile which corresponds to the velocity distributions of two mixing plane jets is determined by the fact that the transverse component in the jet core vanishes. The regularity found for small disturbances is taken as a basis for the further development of the profile from this initial profile.

The transverse component of the flow is determined from the continuity equation, with the use of the approximate function for the velocity component in the main flow direction. For simplicity a linear course of mixing width is assumed.

The dimensions of the mixing region (limiting curve of the jet core $d(x)$ and mixing width $b(x)$) are approximately calculated from the theorem of momentum

$$\left(\bar{u} - u_0\right) vr - \frac{\partial}{\partial x}\int_r^\infty \bar{u}\left(\bar{u} - u_0\right) r \, dr \qquad \left(= r\tau_{xy}\right)$$

$$= r\kappa b(x)\left(u_1 - u_0\right) \frac{\partial \bar{u}}{\partial r}$$

under assumption of a rectilinear course of the velocity distribution over η, where $\eta = \frac{r - d}{b}$.

In order to test the theory by experiment, a measurement was performed for $\frac{u_1 - u_0}{u_1} = 0.5$ with a 5 millimeter nozzle. In order to carry out the comparison with the theory, the conception of the effective nozzle diameter is introduced which complies with the deviation of the effective velocity distribution for an issuing jet from the rectangular velocity distribution

$$u_1\left(u_1 - u_0\right) \frac{r^2_{\text{effect.}}}{2} = \int_0^\infty \bar{u}\left(\bar{u} - u_0\right) r \, dr$$

The agreement between theory and experiment is satisfactory.

V. REFERENCES

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APPENDIX

NO. 1. CALCULATION OF THE TRANSVERSE COMPONENT

$$\bar{v} = -\frac{1}{\left(\eta + \frac{r_0}{x}\right)} \int_{-\eta_k}^{\eta} \left(\eta + \frac{r_0}{x}\right)\left(x\frac{\partial \bar{u}}{\partial x} - \eta\frac{\partial \bar{u}}{\partial \eta}\right) d\eta$$

$$\eta + \frac{r_0}{x} \neq 0$$

or

$$\bar{v} = -\frac{1}{\left(\eta + \frac{r_0}{x}\right)}\left[\left(\frac{x}{r_0}\right)\int_{-\eta_k}^{\eta} \frac{\partial \bar{u}}{\partial \frac{x}{r_0}}\eta\, d\eta + \int_{-\eta_k}^{\eta} \frac{\partial \bar{u}}{\partial \frac{x}{r_0}} d\eta\right] +$$

$$\frac{1}{\left(\eta + \frac{r_0}{x}\right)}\left[\int_{-\eta_k}^{\eta} \frac{\partial \bar{u}}{\partial \eta}\eta^2 d\eta + \left(\frac{r_0}{x}\right)\int_{-\eta_k}^{\eta} \frac{\partial \bar{u}}{\partial \eta}\eta\, d\eta\right]$$

We substitute

$$\frac{\bar{u}}{u_1} = \frac{1}{\sqrt{\pi}}\left(\frac{u_0}{u_1} - 1\right)\int_0^{[\,]} e^{-[\,]^2}\, d[\,] + \frac{1}{2}\left(1 + \frac{u_0}{u_1}\right)$$

$$[\,] = \left(\sigma_1\eta - 0.36\,\frac{u_1 - u_0}{u_1} + \sigma_2\right)$$

$$\frac{\partial \bar{u}/u_1}{\partial \frac{x}{r_0}} = \frac{1}{\sqrt{\pi}}\left(\frac{u_0}{u_1} - 1\right) e^{-[\,]^2}\left[\sigma_1'(\eta + a) + \sigma_2'\right]$$

$$\frac{\partial \bar{u}/u_1}{\partial \eta} = \frac{1}{\sqrt{\pi}}\left(\frac{u_0}{u_1} - 1\right) e^{-[\,]^2}\sigma_1$$

If we assume a rectilinear course of the mixing width $b(x)$, we have $\sigma_1' = 0$.

We then obtain

$$\frac{\bar{v}}{u_1} = -\frac{1}{\left(\eta + \frac{r_0}{x}\right)} \frac{1}{\sqrt{\pi}}\left(\frac{u_0}{u_1} - 1\right) \sigma_2' \left[\left(\frac{x}{r_0}\right)\int_{-\eta_k}^{\eta} e^{-[\,]^2} \eta \, d\eta + \int_{-\eta_k}^{\eta} e^{-[\,]^2} d\eta\right] +$$

$$\frac{1}{\left(\eta + \frac{r_0}{x}\right)} \frac{1}{\sqrt{\pi}}\left(\frac{u_0}{u_1} - 1\right) \sigma_1\left[\int_{-\eta_k}^{\eta} e^{-[\,]^2} \eta^2 d\eta + \frac{r_0}{x}\int_{-\eta_k}^{\eta} e^{-[\,]^2} \eta \, d\eta\right]$$

The evaluation of the integrals yields

(a)

$$\int_{-\eta_k}^{\eta} e^{-[\,]^2} d\eta$$

$$\int e^{-[\,]^2} d\eta = \frac{1}{\sigma_1}\int e^{-[\,]^2} d(\sigma_1\eta) = \frac{1}{\sigma_1}\int e^{-[\,]^2} d[\,]$$

$$[\,] = \left(\sigma_1\eta - 0.36 \frac{u_1 - u_0}{u_1} + \sigma_2\right)$$

thus

$$\int_{-\eta_k}^{\eta} e^{-[\,]^2} d\eta = \frac{1}{\sigma_1}\int_{[\,]_0}^{[\,]} e^{-[\,]^2} d[\,]$$

$$[\,]_0 = \left[\sigma_1(-\eta_k) - 0.36 \frac{u_1 - u_0}{u_1} + \sigma_2\right]$$

$$\int_{\square_0}^{\square} = \int_{\square_0}^{0} + \int_{0}^{\square} = \int_{0}^{-\square_0} + \int_{0}^{\square}$$

$$\int_{0}^{\square} e^{-\square^2}\, d\square = \frac{\sqrt{\pi}}{2}\Big[F_1{}'\Big]_{\square}$$

where

$$F_1{}' = \frac{2}{\sqrt{\pi}}\int_{0}^{\square} e^{-\square^2}\, d\square_{\text{error integral}}$$

thus

$$\int_{-\eta_k}^{\eta} e^{-\square^2}\, d\eta = \frac{1}{\sigma_1}\,\frac{\sqrt{\pi}}{2}\left(F_1{}'_{\square} + F_1{}'_{-\square_0}\right)$$

(b)

$$\int_{-\eta_k}^{\eta} e^{-\square^2}\, \eta\, d\eta$$

$$\square = \sigma_1\eta - 0.36\,\frac{u_1 - u_0}{u_1} + \sigma_2$$

$$\eta = \frac{\square + 0.36\,\dfrac{u_1 - u_0}{u_1} - \sigma_2}{\sigma_1}$$

This results in

$$\int e^{-\square^2}\, \eta\, d\eta = \frac{1}{\sigma_1}\int e^{-\square^2}\, \frac{\square + 0.36\,\dfrac{u_1 - u_0}{u_1} - \sigma_2}{\sigma_1}\, d(\sigma_1\eta)$$

$$= \frac{1}{\sigma_1{}^2}\int e^{-\square^2}\, \square\, d\square - \left(\frac{\sigma_2 - 0.36\,\dfrac{u_1 - u_0}{u_1}}{\sigma_1{}^2}\right)\int e^{-\square^2}\, d\square$$

$$\int_{-\eta_k}^{\eta} e^{-[\,]^2}\,\eta\,d\eta = \frac{1}{\sigma_1{}^2}\int_{[\,]_0}^{[\,]} e^{-[\,]^2}\,[\,]\,d[\,] - \left(\frac{\sigma_2 - 0.36\,\frac{u_1 - u_0}{u_1}}{\sigma_1{}^2}\right)\int_{[\,]_0}^{[\,]} e^{-[\,]^2}\,d[\,]$$

$$= \frac{1}{\sigma_1{}^2}\left(-\frac{1}{2}\,e^{-[\,]^2}\right)_{[\,]_0}^{[\,]} - \left(\frac{\sigma_2 - 0.36\,\frac{u_1 - u_0}{u_1}}{\sigma_1{}^2}\right)\frac{\sqrt{\pi}}{2}\left(F_1{}'_{[\,]} + F_1{}'_{-[\,]_0}\right)$$

$$\int_{-\eta_k}^{\eta} e^{-[\,]^2}\,\eta\,d\eta = \frac{1}{\sigma_1{}^2}\left(-\frac{1}{2}\,e^{-[\,]^2} + \frac{1}{2}\,e^{-[\,]_0{}^2}\right) - \left(\frac{\sigma_2 - 0.36\,\frac{u_1 - u_0}{u_1}}{\sigma_1{}^2}\right)\frac{\sqrt{\pi}}{2}\left(F_1{}'_{[\,]} + F_1{}'_{-[\,]_0}\right)$$

(c)

$$\int_{-\eta_k}^{\eta} e^{-[\,]^2}\,\eta^2 d\eta$$

$$\int e^{-[\,]^2}\,\eta^2 d\eta = \frac{1}{\sigma_1}\int e^{-[\,]^2}\left[\frac{[\,] - \left(\sigma_2 - 0.36\,\frac{u_1 - u_0}{u_1}\right)}{\sigma_1}\right]^2 d(\sigma_1\eta)$$

$$= \frac{1}{\sigma_1{}^3}\int e^{-[\,]^2}\,[\,]^2\,d[\,] - \frac{2}{\sigma_1{}^3}\left(\sigma_2 - 0.36\,\frac{u_1 - u_0}{u_1}\right)\int e^{-[\,]^2}\,[\,]\,d[\,] +$$

$$\frac{\left(\sigma_2 - 0.36\,\frac{u_1 - u_0}{u_1}\right)^2}{\sigma_1{}^3}\int e^{-[\,]^2}\,d[\,]$$

$$\int e^{-[\,]^2} [\,]^2 \, d[\,] = \frac{1}{2} [\,] \, e^{-[\,]^2} + \frac{1}{2}\int e^{-[\,]^2} \, d[\,]$$

thus

$$\int_{-\eta_k}^{\eta} e^{-[\,]^2} \eta^2 d\eta = \frac{1}{\sigma_1{}^3}\left(-\frac{1}{2} [\,] \, e^{-[\,]^2}\right)_{[\,]_0}^{[\,]} + \frac{1}{\sigma_1{}^3}\,\frac{1}{2}\int_{[\,]_0}^{[\,]} e^{-[\,]^2} \, d[\,] - \frac{2}{\sigma_1{}^3}\Bigg(\sigma_2 -$$

$$0.36 \, \frac{u_1 - u_0}{u_1}\Bigg)\left(-\frac{1}{2} [\,] \, e^{-[\,]^2}\right)_{[\,]_0}^{[\,]} + \frac{\left(\sigma_2 - 0.36 \, \frac{u_1 - u_0}{u_1}\right)^2}{\sigma_1{}^3} \, \frac{\sqrt{\pi}}{2}\left(F_1{}'_{[\,]} + F_1{}'_{-[\,]_0}\right)$$

$$\int_{-\eta_k}^{\eta} e^{-[\,]^2} \eta^2 d\eta = \frac{1}{2}\,\frac{1}{\sigma_1{}^3}\left(-[\,] \, e^{-[\,]^2} + [\,]_0 \, e^{-[\,]_0{}^2}\right) + \frac{\left(\sigma_2 - 0.36 \, \frac{u_1 - u_0}{u_1}\right)}{\sigma_1{}^3}\left(e^{-[\,]^2} - e^{-[\,]_0{}^2}\right) +$$

$$\frac{\sqrt{\pi}}{2}\left(F_1{}'_{[\,]} + F_1{}'_{-[\,]_0}\right) \frac{1}{\sigma_1{}^3}\left[\frac{1}{2} + \left(\sigma_2 - 0.36 \, \frac{u_1 - u_0}{u_1}\right)^2\right]$$

If one substitutes these expressions, one obtains for $\frac{\bar{v}}{u_1}$ the formula given in the text.

APPENDIX

NO. 2. FOR CALCULATION OF THE DIMENSIONS OF THE CORE REGION

The theorem of momentum with $r = r_0$ reads

$$\left(\frac{\bar{u} - u_0}{u_1}\right)_{r=r_0} \left(\frac{\bar{v}}{u_1}\right)_{r=r_0} r_0 - \frac{\partial}{\partial x}\left[\int_{r_0}^{drb} \frac{\bar{u}}{u_1}\left(\frac{\bar{u}_1 - u_0}{u_1}\right) r\, dr\right] =$$

$$r_0 \kappa b \left(\frac{u_1 - u_0}{u_1}\right)\left(\frac{\partial \frac{\bar{u}}{u_1}}{\partial r}\right)_{r=r_0}$$

With the coordinate transformation $\eta = \frac{r - d}{b}$ we obtain

$$\left(\frac{\bar{v}}{u_1}\right)_{r=r_0} = -\frac{1}{r_0}\int_0^{\frac{r_0-d}{b}} \frac{d\, \bar{u}/u_1}{d\eta}\left[-\eta^2 bb' - \eta(bd' + b'd) - dd'\right] d\eta$$

$$\frac{\partial}{\partial x}\left(\int_{r_0}^{d+b}\right) = \int_{\frac{r_0-d}{b}}^{1} \left\{\frac{d}{d\eta}\left[\frac{\bar{u}}{u_1}\left(\frac{\bar{u} - u_0}{u_1}\right)\right]\right\}\left[-\eta^2 bb' - \eta(bd' + b'd) - dd'\right] d\eta$$

We substitute

$$\frac{\bar{u} - u_0}{u_1} = \left(\frac{u_1 - u_0}{u_1}\right)(1 - \eta)$$

$$\frac{d\left(\frac{\bar{u}}{u_1}\right)}{d\eta} = -\left(\frac{u_1 - u_0}{u_1}\right)$$

$$\frac{d}{d\eta}\left[\frac{\bar{u}}{u_1}\left(\frac{\bar{u} - u_0}{u_1}\right)\right] = - 2\left(\frac{u_1 - u_0}{u_1}\right)^2 (1 - \eta) - \frac{u_0}{u_1}\left(\frac{u_1 - u_0}{u_1}\right)$$

$$\frac{\partial\left(\frac{\bar{u}}{u_1}\right)}{\partial r} = - \frac{1}{b}\left(\frac{u_1 - u_0}{u_1}\right)$$

This results in

$$\left(\frac{\bar{u} - u_0}{u_1}\right)\left(\frac{\bar{v}}{u_1}\right)_{r=r_0} r_0 = -\left(\frac{u_1 - u_0}{u_1}\right)\left(1 - \frac{r_0 - d}{b}\right)\left(\frac{u_1 - u_0}{u_1}\right)\left[bb'\int_0^\eta \eta^2 d\eta + (bd' + b'd)\int_0^\eta \eta\ d\eta + dd'\int_0^\eta d\eta\right]$$

$$\frac{\partial}{\partial x}\left[\int_{r_0}^{d+b} \frac{\bar{u}}{u_1}\left(\frac{\bar{u} - u_0}{u_1}\right) r\ dr\right] = \int_{\frac{r_0-d}{b}}^{1} \left[2\left(\frac{u_1 - u_0}{u_1}\right)^2 (1 - \eta) + \frac{u_0}{u_1}\left(\frac{u_1 - u_0}{u_1}\right)\right]\left[\eta^2 bb' + \eta(bd' + b'd) + dd'\right] d\eta$$

$$r_0 \kappa b\left(\frac{u_1 - u_0}{u_1}\right)\frac{\partial\left(\frac{\bar{u}}{u_1}\right)}{\partial r} = - r_0 \kappa\left(\frac{u_1 - u_0}{u_1}\right)^2$$

If one evaluates the integrals, one obtains

$$\left(\frac{\bar{u} - u_0}{u_1}\right)\left(\frac{\bar{v}}{u_1}\right)_{r=r_0} r_0 = -\left(\frac{u_1 - u_0}{u_1}\right)^2 \left(1 - \frac{r_0 - d}{b}\right)\left[\frac{bb'}{3}\left(\frac{r_0 - d}{b}\right)^3 + \left(\frac{bd' + b'd}{2}\right)\left(\frac{r_0 - d}{b}\right)^2 + dd'\left(\frac{r_0 - d}{b}\right)\right]$$

$$\frac{\partial}{\partial x}\left[\int_r^{d+b} \frac{\bar{u}}{u_1}\left(\frac{\bar{u} - u_0}{u_1}\right) r\, dr\right] = bb'\left[2\left(\frac{u_1 - u_0}{u_1}\right)^2 \int_{\frac{r_0-d}{b}}^1 \eta^2(1 - \eta)d\eta + \frac{u_0}{u_1}\left(\frac{u_1 - u_0}{u_1}\right)\int_{\frac{r_0-d}{b}}^1 \eta^2 d\eta\right] + (b'd + bd')\left[2\left(\frac{u_1 - u_0}{u_1}\right)^2 \int_{\frac{r_0-d}{b}}^1 \eta(1 - \eta)d\eta + \frac{u_0}{u_1}\left(\frac{u_1 - u_0}{u_1}\right)\int_{\frac{r_0-d}{b}}^1 \eta\, d\eta\right] + dd'\left[2\left(\frac{u_1 - u_0}{u_1}\right)^2 \int_{\frac{r_0-d}{b}}^1 (1 - \eta)d\eta + \frac{u_0}{u_1}\left(\frac{u_1 - u_0}{u_1}\right)\int_{\frac{r_0-d}{b}}^1 d\eta\right] = bb'\left\{2\left(\frac{u_1 - u_0}{u_1}\right)^2\left[\frac{1}{12} - \frac{1}{3}\left(\frac{r_0 - d}{b}\right)^3 + \frac{1}{4}\left(\frac{r_0 - d}{b}\right)^4\right] + \frac{u_0}{u_1}\left(\frac{u_1 - u_0}{u_1}\right)\left[\frac{1}{3} - \frac{1}{3}\left(\frac{r_0 - d}{b}\right)^3\right]\right\} + (b'd + bd')\left\{2\left(\frac{u_1 - u_0}{u_1}\right)^2\left[\frac{1}{6} - \frac{1}{2}\left(\frac{r_0 - d}{b}\right)^2 + \frac{1}{3}\left(\frac{r_0 - d}{b}\right)^3\right] + \frac{u_0}{u_1}\left(\frac{u_1 - u_0}{u_1}\right)\left[\frac{1}{2} - \frac{1}{2}\left(\frac{r_0 - d}{b}\right)^2\right]\right\} + dd'\left\{2\left(\frac{u_1 - u_0}{u_1}\right)^2\left[\frac{1}{2} - \left(\frac{r_0 - d}{b}\right) + \frac{1}{2}\left(\frac{r_0 - d}{b}\right)^2\right] + \frac{u_0}{u_1}\left(\frac{u_1 - u_0}{u_1}\right)\left[1 - \left(\frac{r_0 - d}{b}\right)\right]\right\}$$

If we insert these expressions into the equation of momentum and order, we obtain

$$bb'\left[-\frac{1}{3}\left(\frac{u_1-u_0}{u_1}\right)^2\left(\frac{r_0-d}{b}\right)^3+\frac{1}{6}\left(\frac{u_1-u_0}{u_1}\right)^2\left(\frac{r_0-d}{b}\right)^4-\frac{1}{6}\left(\frac{u_1-u_0}{u_1}\right)^2+\right.$$

$$\frac{2}{3}\left(\frac{u_1-u_0}{u_1}\right)^2\left(\frac{r_0-d}{b}\right)^3-\frac{1}{2}\left(\frac{u_1-u_0}{u_1}\right)^2\left(\frac{r_0-d}{b}\right)^4-\frac{1}{3}\frac{u_0}{u_1}\left(\frac{u_1-u_0}{u_1}\right)+$$

$$\left.\frac{1}{3}\frac{u_0}{u_1}\left(\frac{u_1-u_0}{u_1}\right)\left(\frac{r_0-d}{b}\right)^3\right]+(bd'+b'd)\left[-\frac{1}{2}\left(\frac{u_1-u_0}{u_1}\right)^2\left(\frac{r_0-d}{b}\right)^2+\right.$$

$$\frac{1}{2}\left(\frac{u_1-u_0}{u_1}\right)^2\left(\frac{r_0-d}{b}\right)^3-\frac{1}{3}\left(\frac{u_1-u_0}{u_1}\right)^2+\left(\frac{u_1-u_0}{u_1}\right)^2\left(\frac{r_0-d}{b}\right)^2-$$

$$\left.\frac{2}{3}\left(\frac{u_1-u_0}{u_1}\right)^2\left(\frac{r_0-d}{b}\right)^3-\frac{1}{2}\frac{u_0}{u_1}\left(\frac{u_1-u_0}{u_1}\right)+\frac{1}{2}\frac{u_0}{u_1}\left(\frac{u_1-u_0}{u_1}\right)\left(\frac{r_0-d}{b}\right)^2\right]+$$

$$dd'\left[-\left(\frac{u_1-u_0}{u_1}\right)^2\left(\frac{r_0-d}{b}\right)+\left(\frac{u_1-u_0}{u_1}\right)^2\left(\frac{r_0-d}{b}\right)^2-\left(\frac{u_1-u_0}{u_1}\right)^2+\right.$$

$$2\left(\frac{u_1-u_0}{u_1}\right)^2\left(\frac{r_0-d}{b}\right)-\left(\frac{u_1-u_0}{u_1}\right)\left(\frac{r_0-d}{b}\right)^2-\frac{u_0}{u_1}\left(\frac{u_1-u_0}{u_1}\right)+$$

$$\left.\frac{u_0}{u_1}\left(\frac{u_1-u_0}{u_1}\right)\left(\frac{r_0-d}{b}\right)\right]=-r_0\kappa\left(\frac{u_1-u_0}{u_1}\right)^2$$

which finally leads to

$$bb'\left\{\left(\frac{u_1 - u_0}{u_1}\right)\left[-\frac{1}{6}\left(\frac{r_0 - d}{b}\right)^4 + \frac{1}{6}\right] + \frac{1}{3}\left(\frac{r_0 - d}{b}\right)^3 - \frac{1}{3}\right\} +$$

$$(b'd + bd')\left\{\left(\frac{u_1 - u_0}{u_1}\right)\left[-\frac{1}{6}\left(\frac{r_0 - d}{b}\right)^3 + \frac{1}{6}\right] + \frac{1}{2}\left(\frac{r_0 - d}{b}\right)^2 - \frac{1}{2}\right\} +$$

$$dd'\left[\left(\frac{r_0 - d}{b}\right) - 1\right] = - r_0\kappa\left(\frac{u_1 - u_0}{u_1}\right)$$

Translated by Mary L. Mahler
National Advisory Committee
for Aeronautics

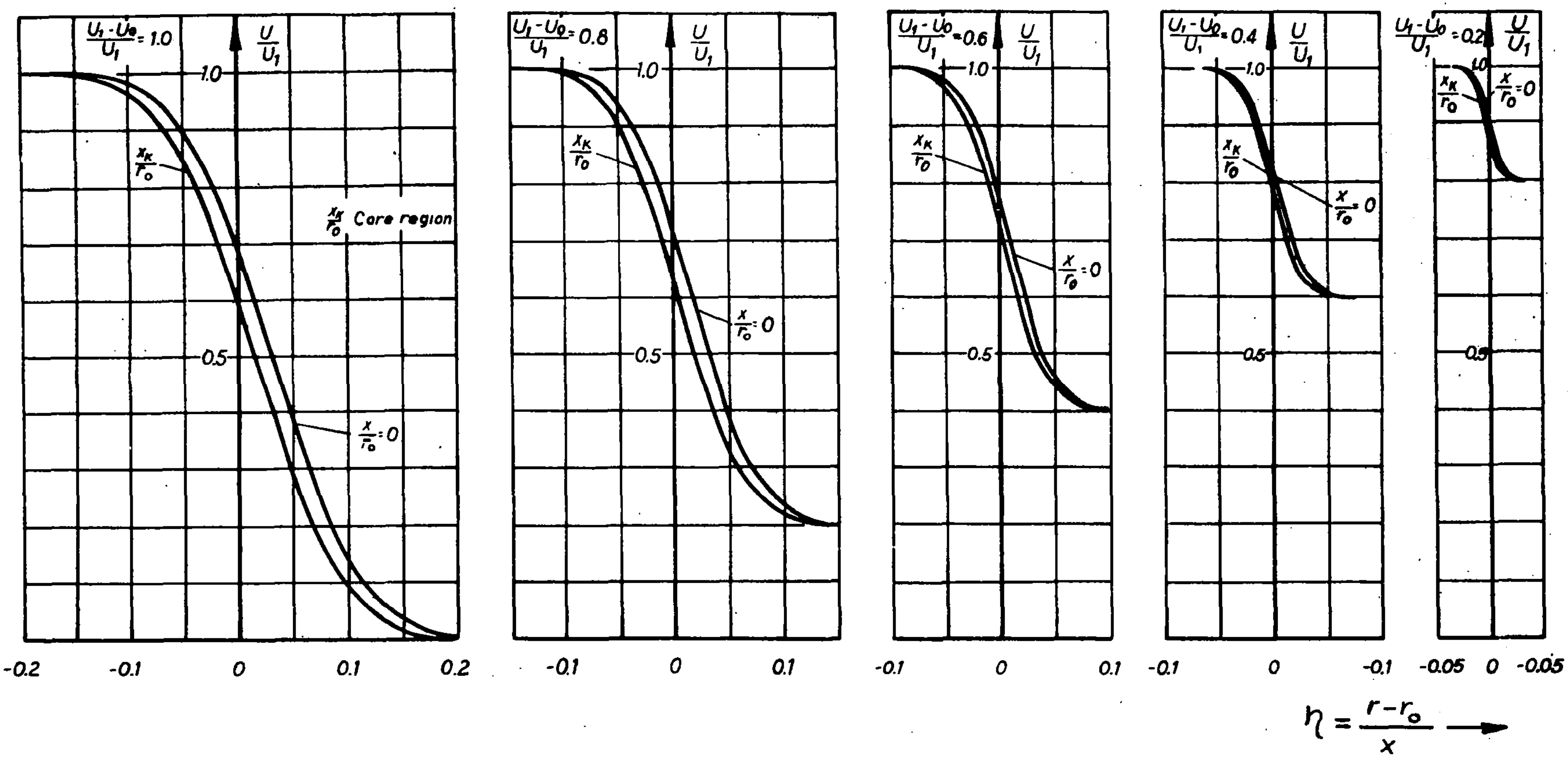


Figure 1.- Velocity distributions.

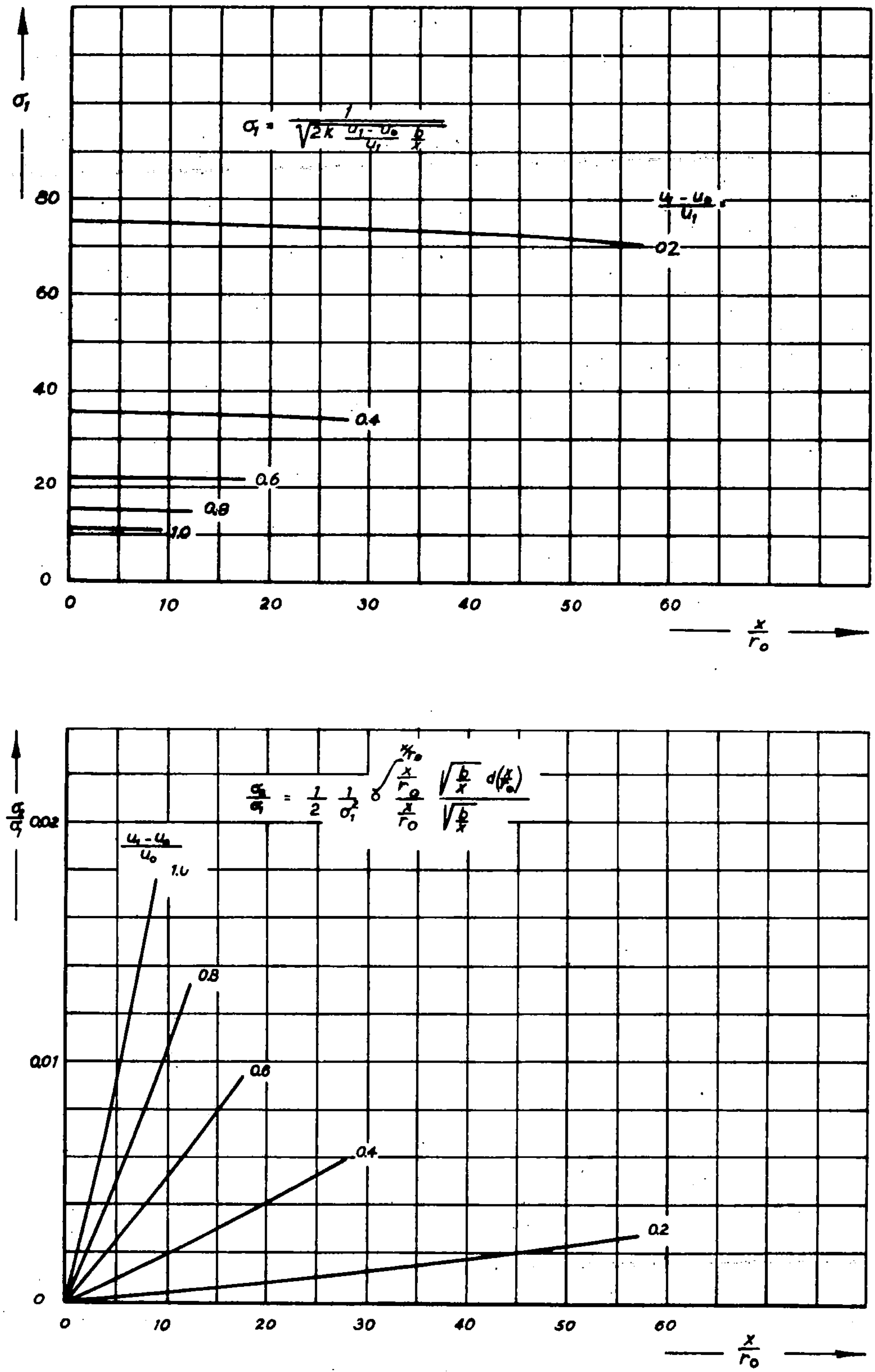


Figure 2.- Auxiliary quantities for calculation of the velocity distribution.

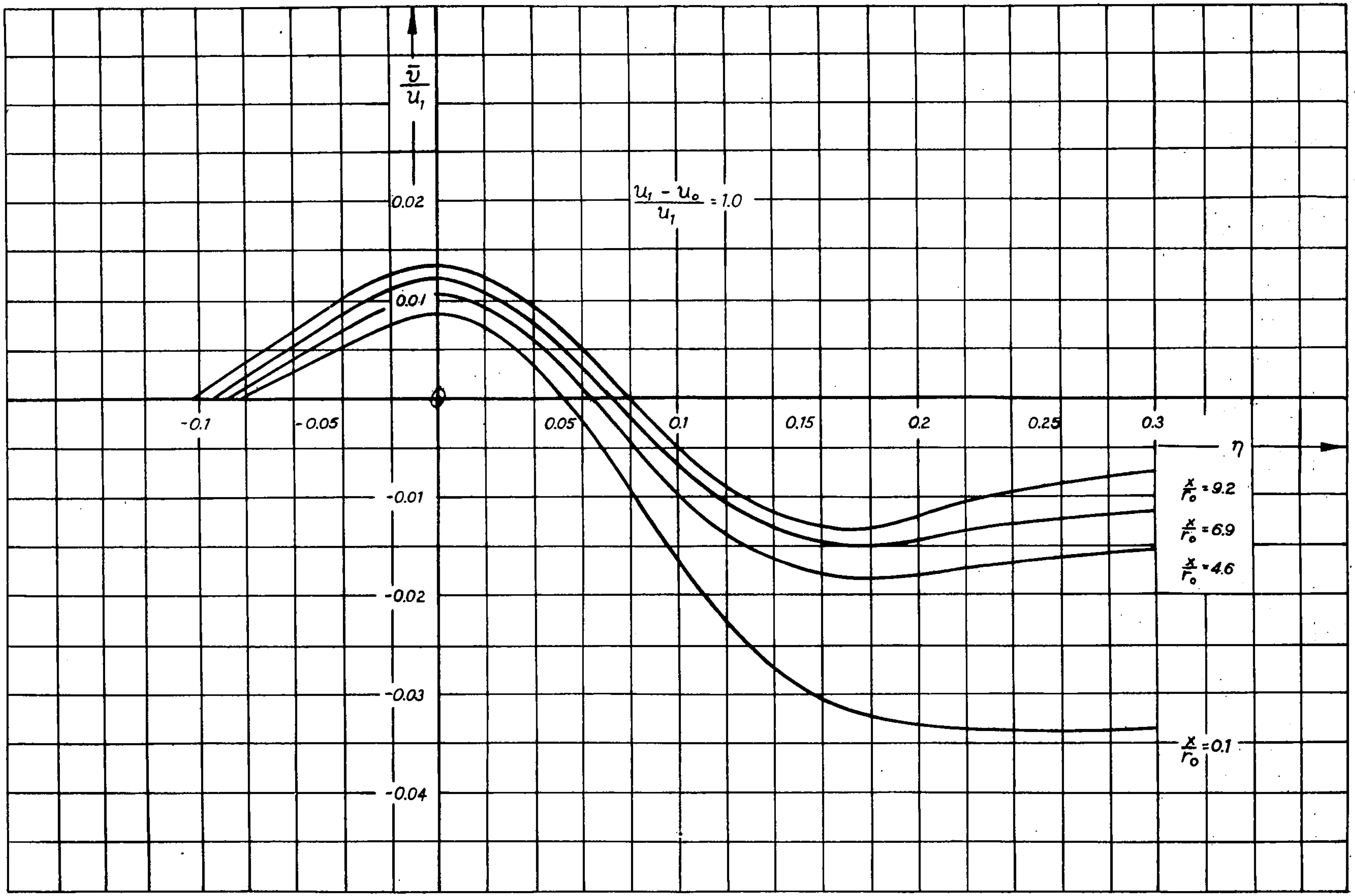


Figure 3.- Transverse component distribution.

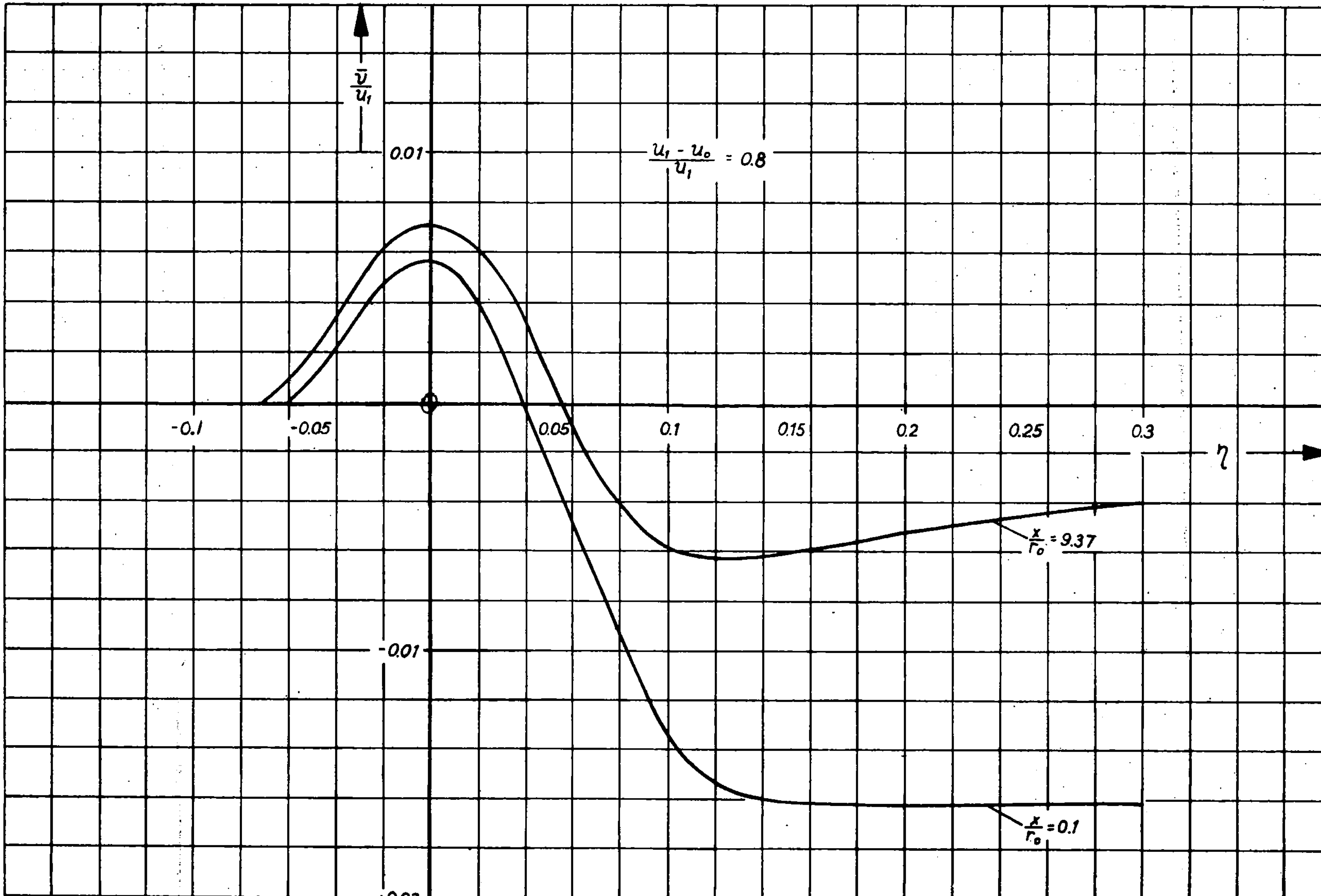


Figure 4.- Transverse component distribution.

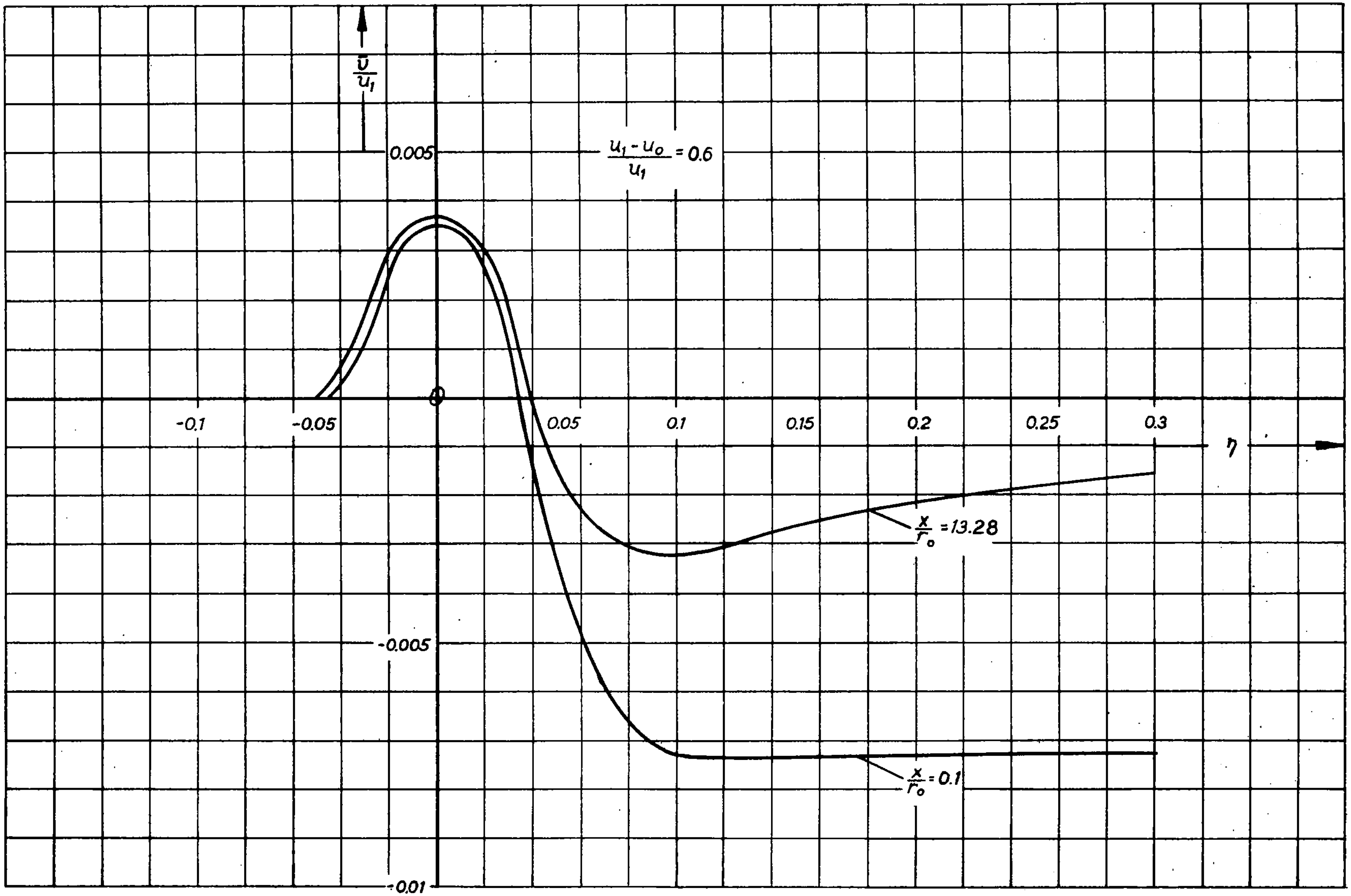


Figure 5.- Transverse component distribution.

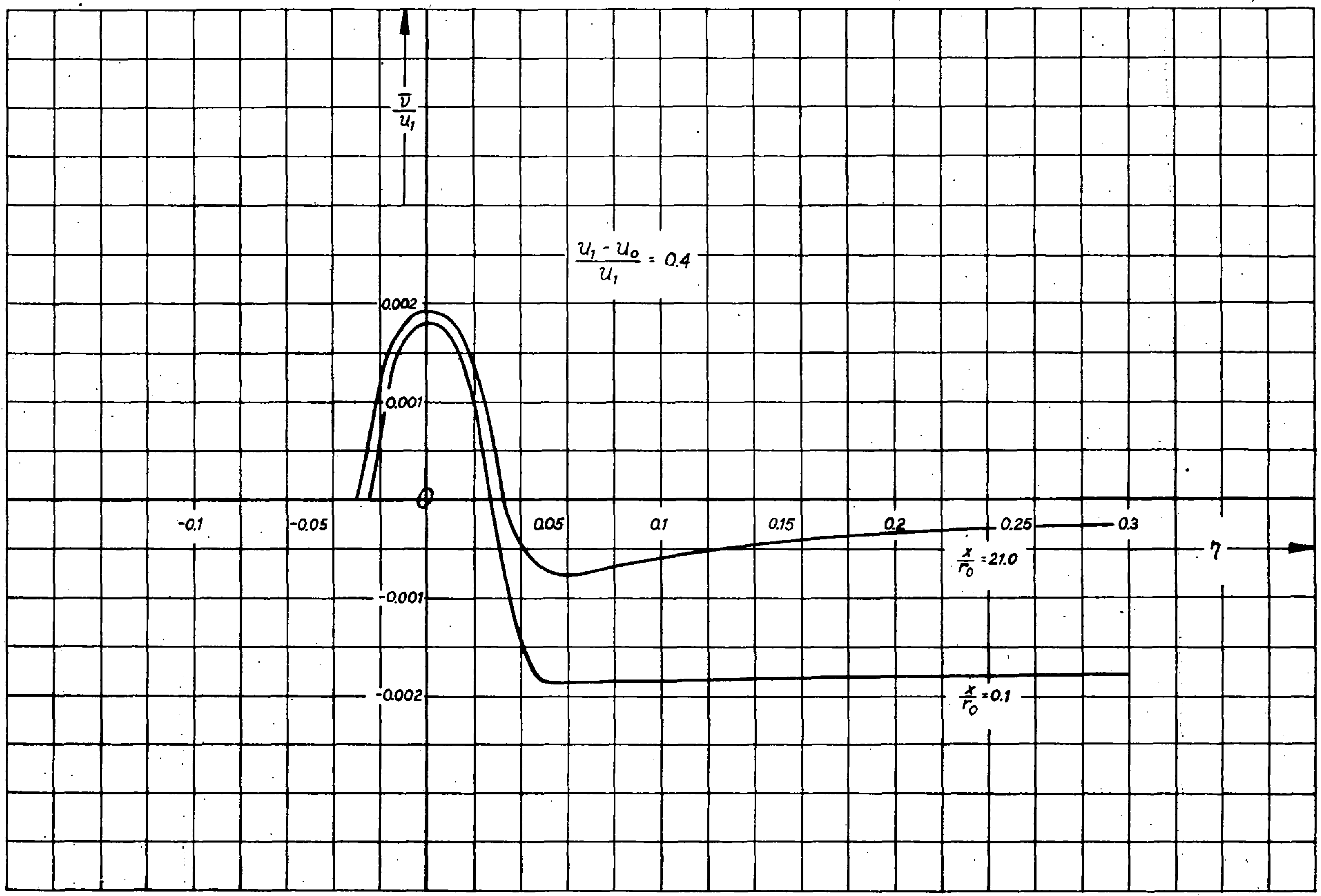


Figure 6.- Transverse component distribution.

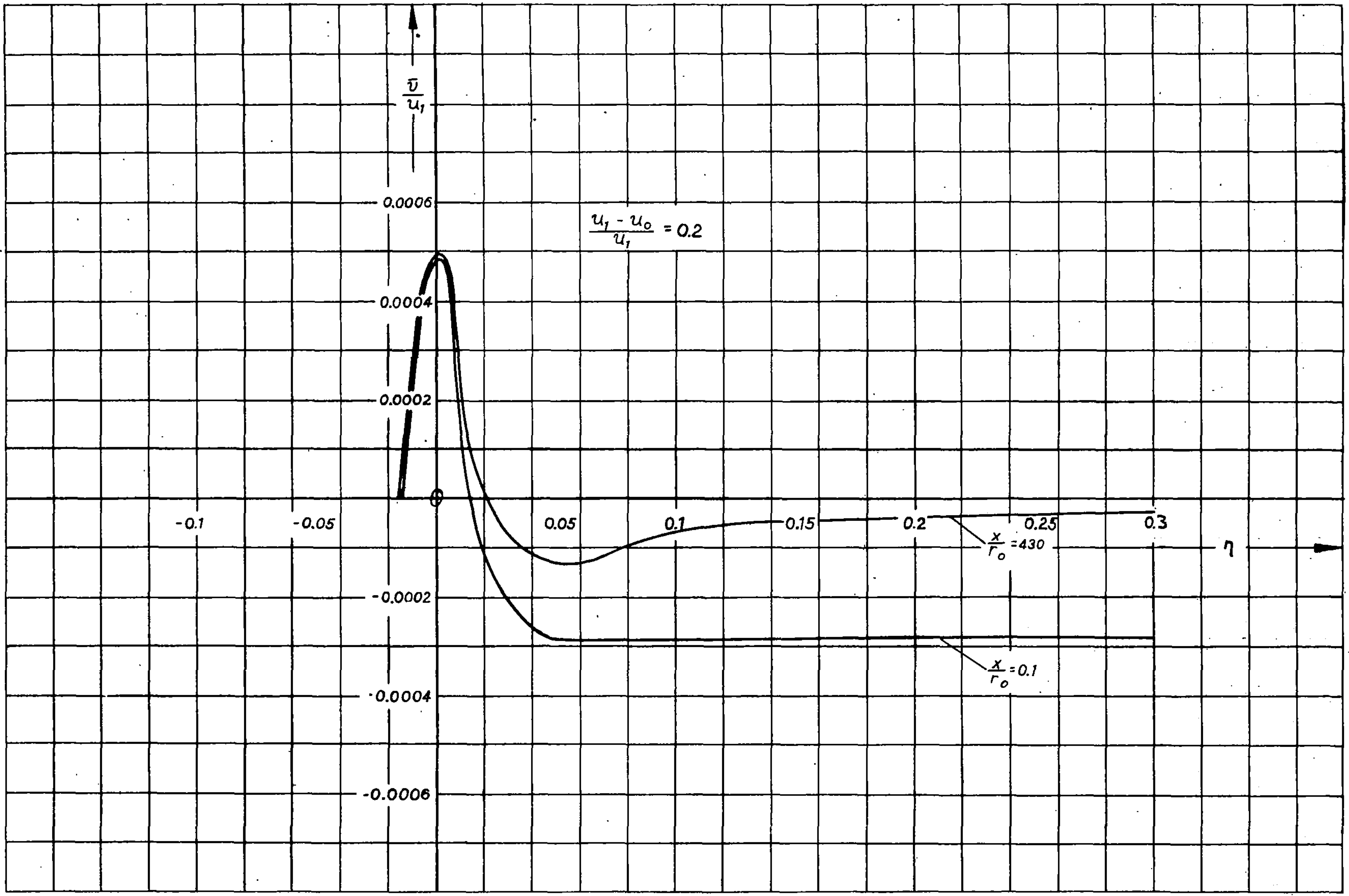


Figure 7.- Transverse component distribution.

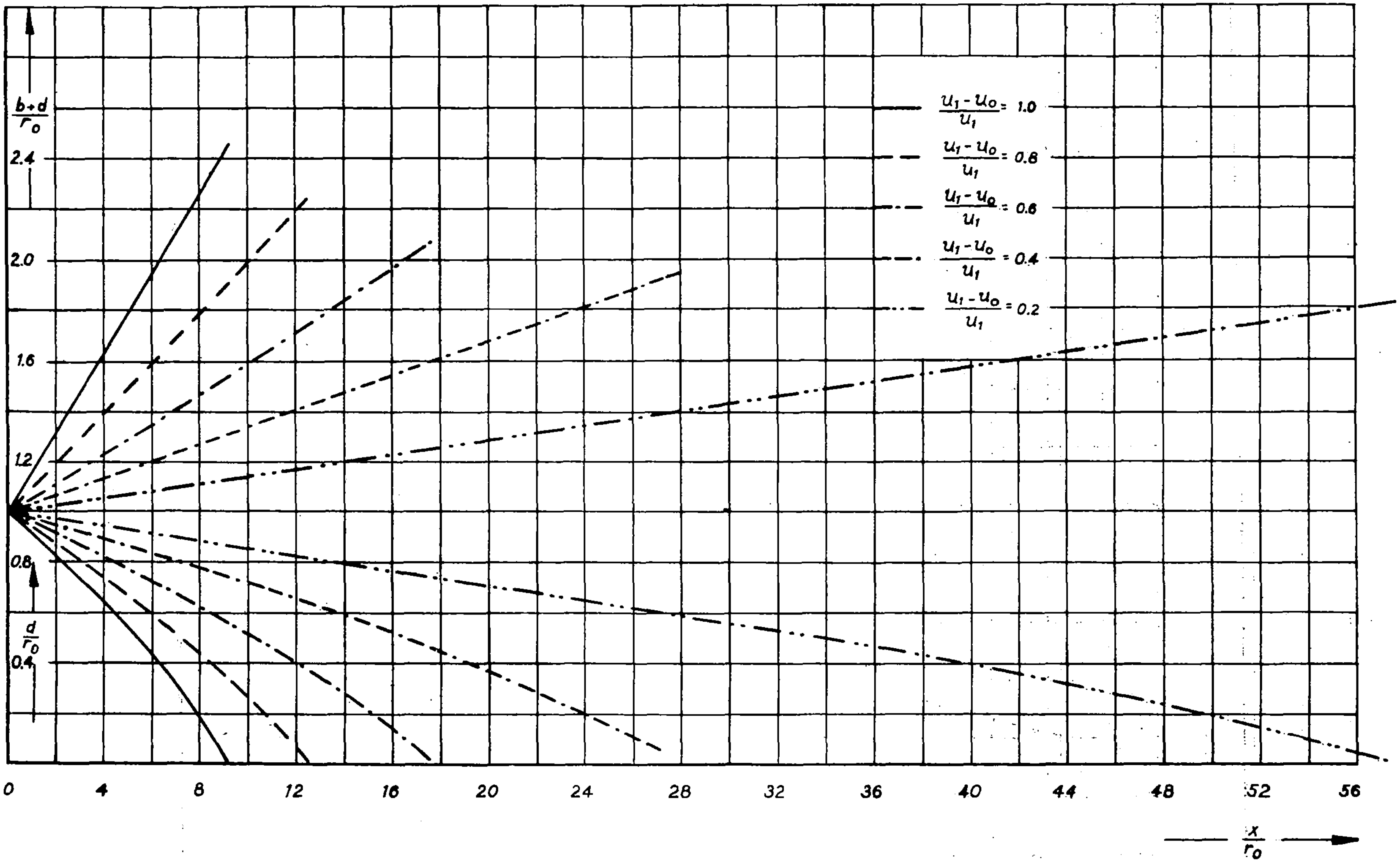


Figure 8.- Dimensions of the core region.

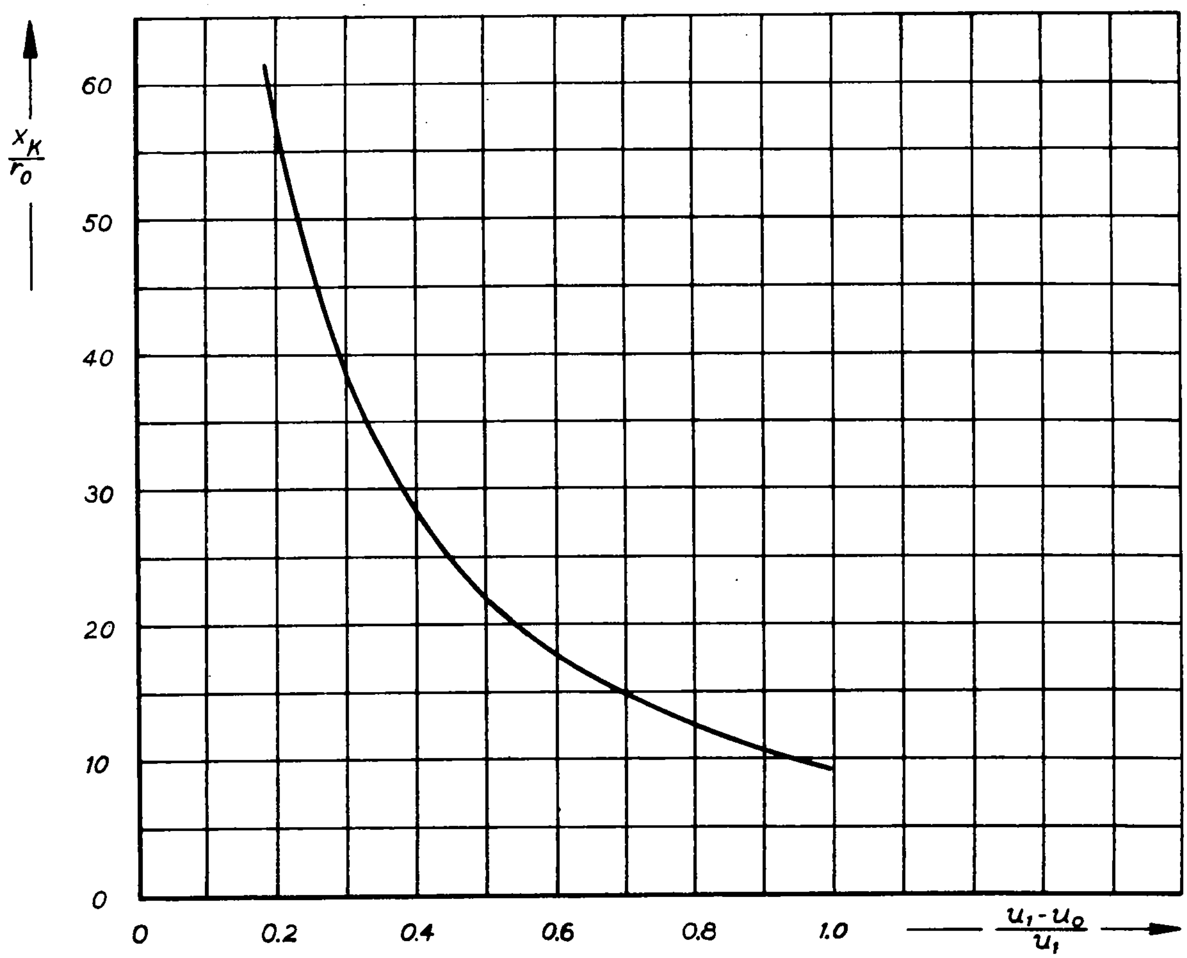


Figure 9.- Core lengths.

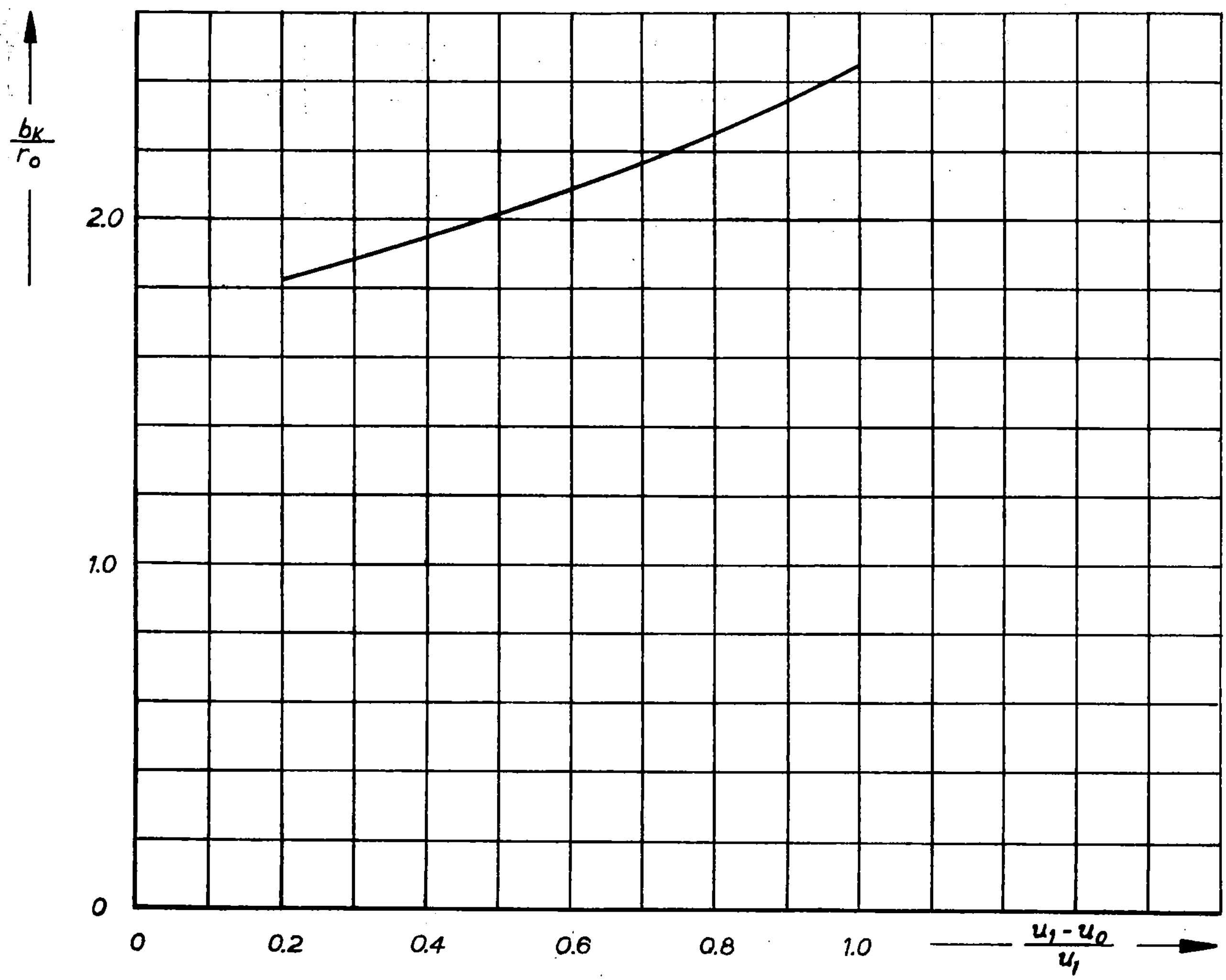


Figure 10.- Mixing widths at the core end.

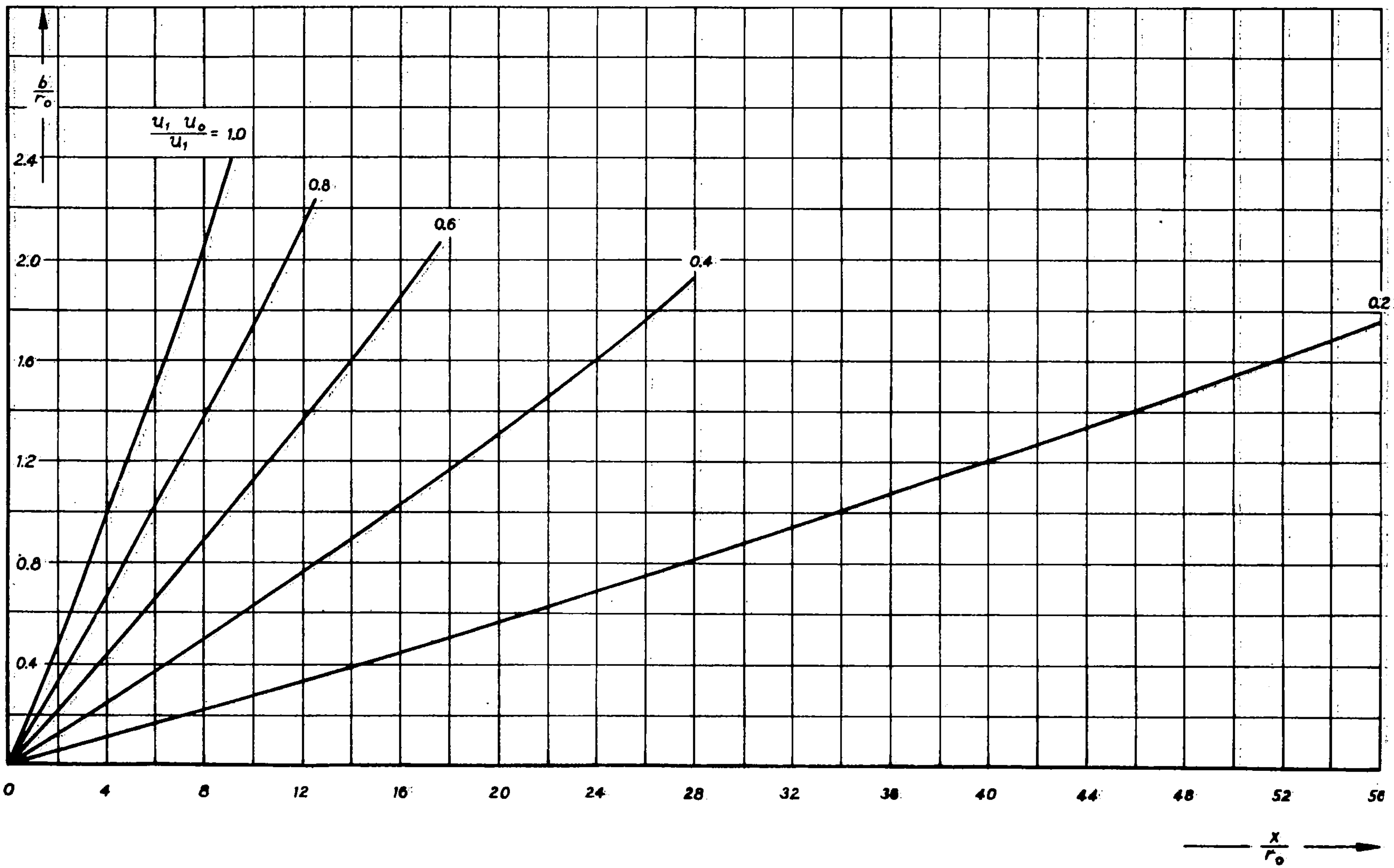


Figure 11.- Mixing width variation.

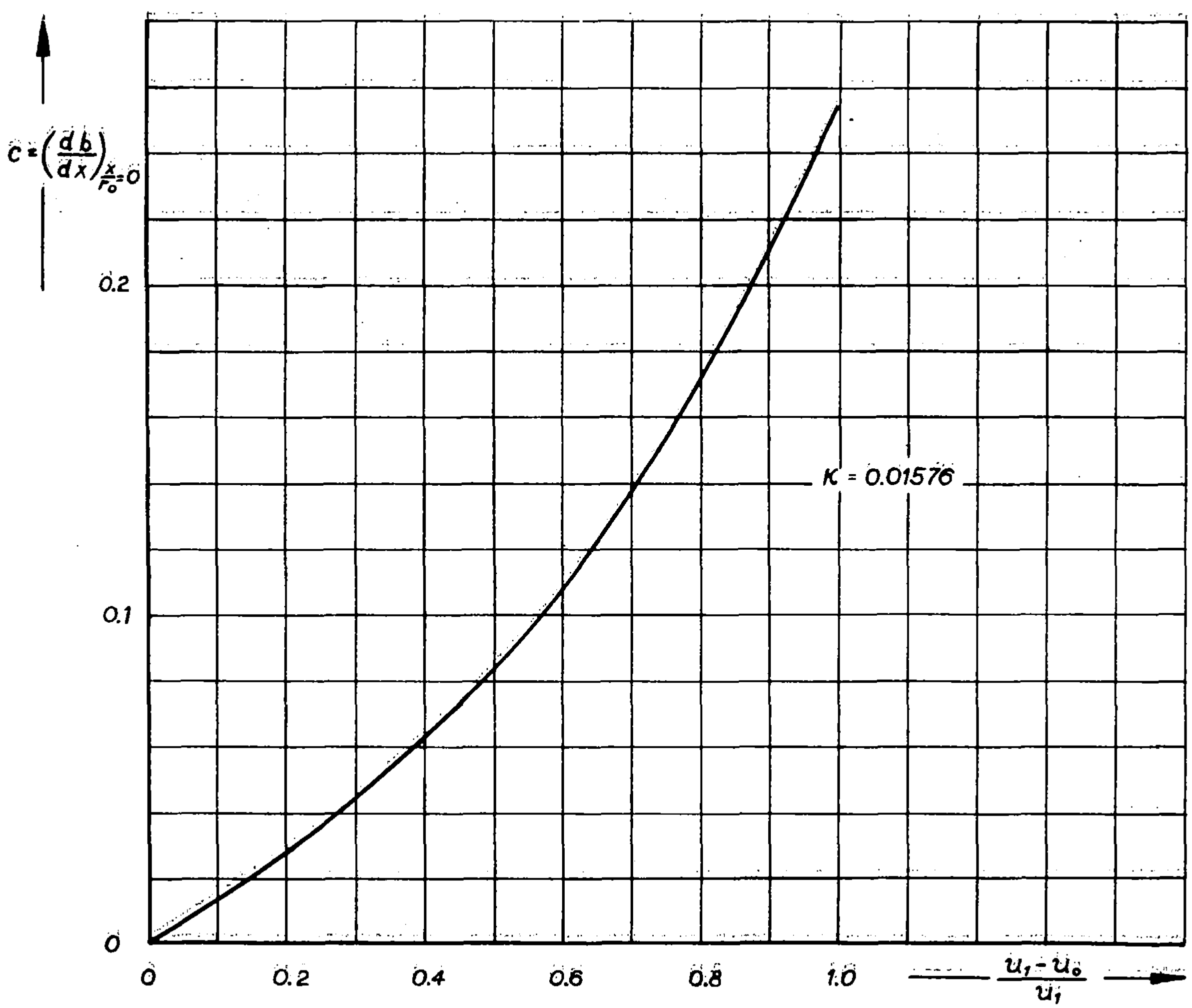


Figure 12.- Angle of spread of the mixing region.

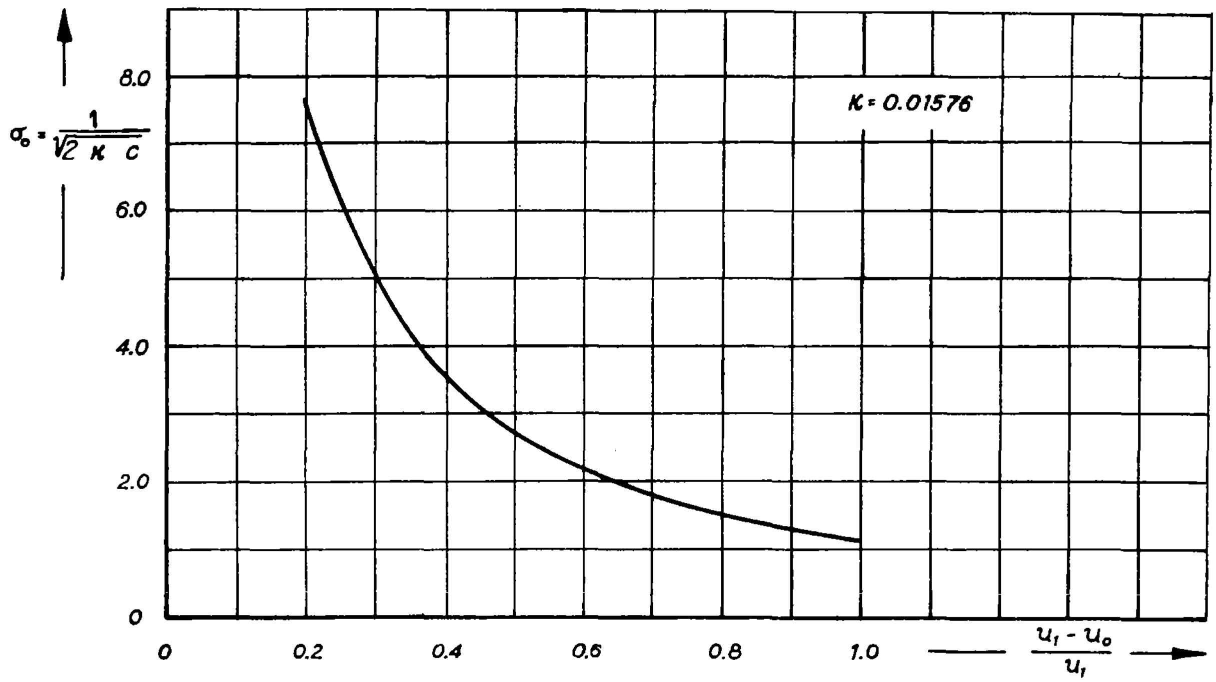


Figure 13.- Parameter value of the plane jet rim.

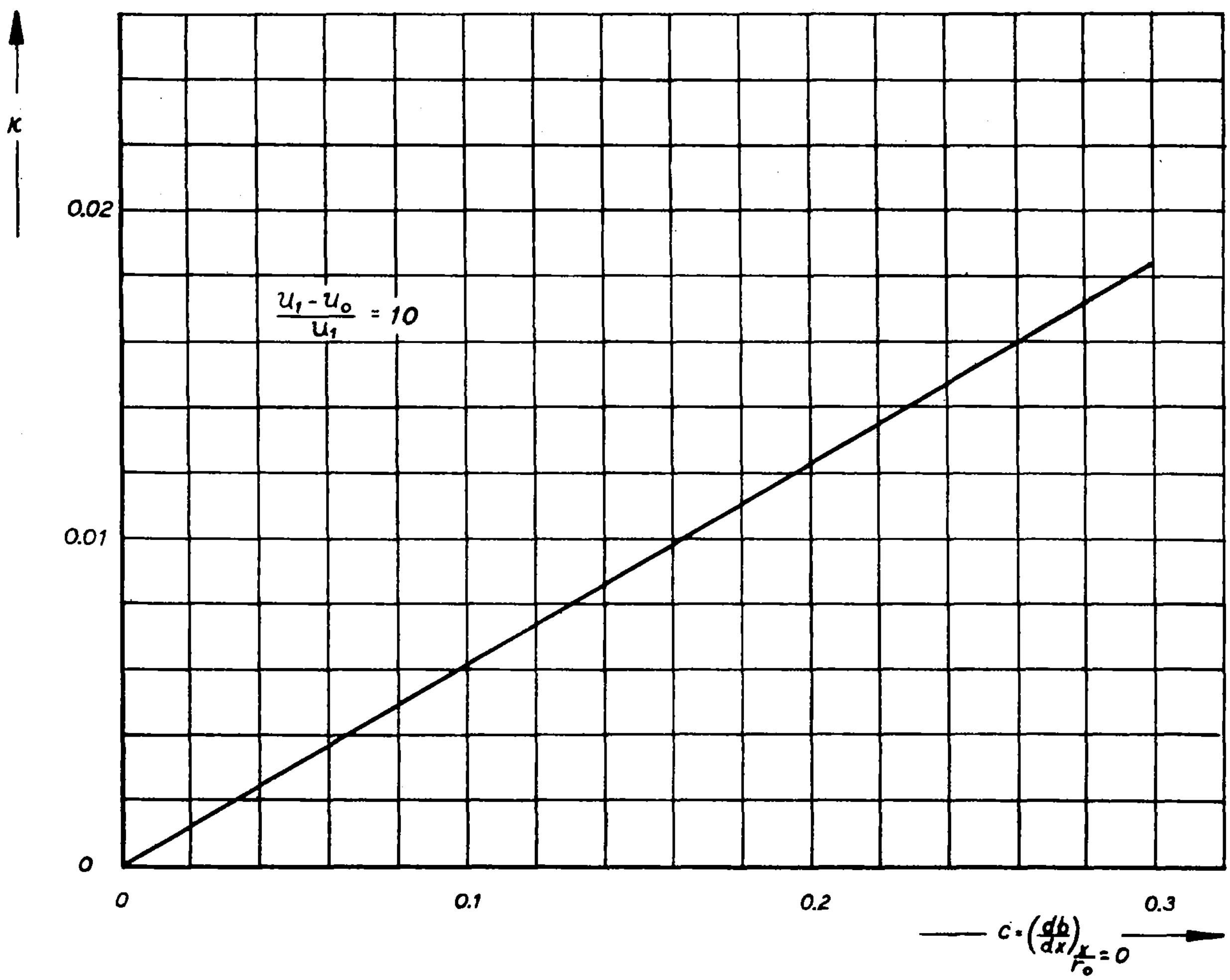


Figure 14.- κ as a function of c.

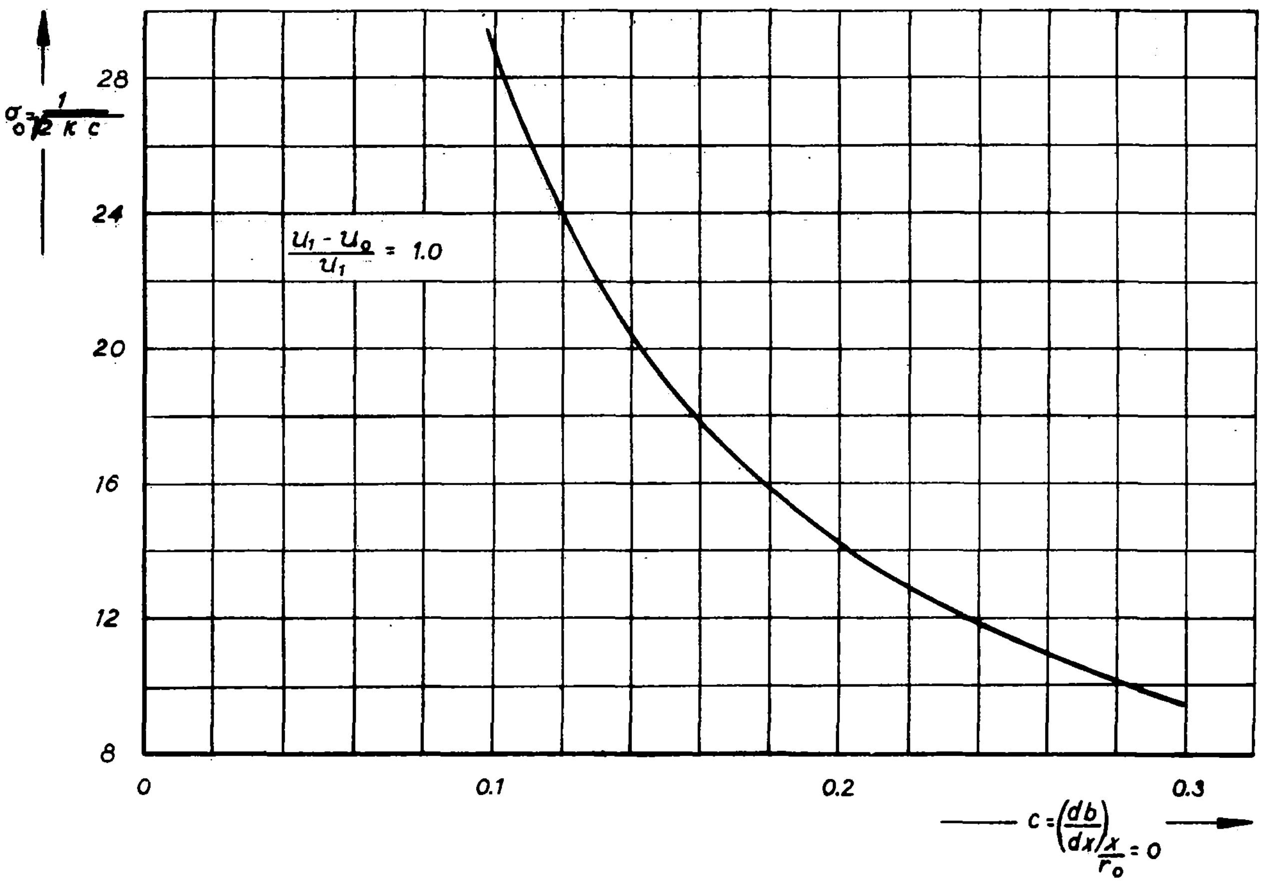


Figure 15.- σ_0 as a function of c.

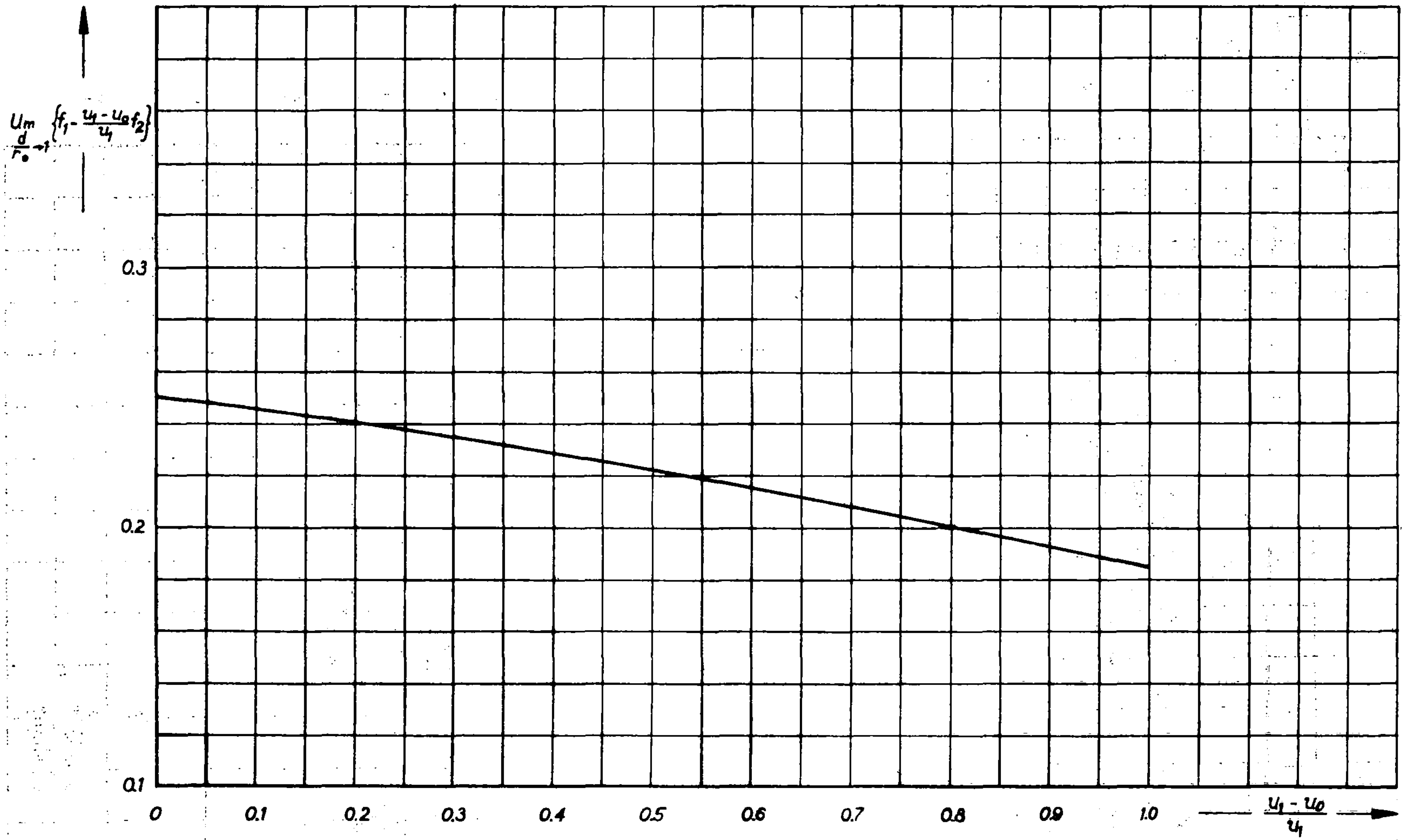
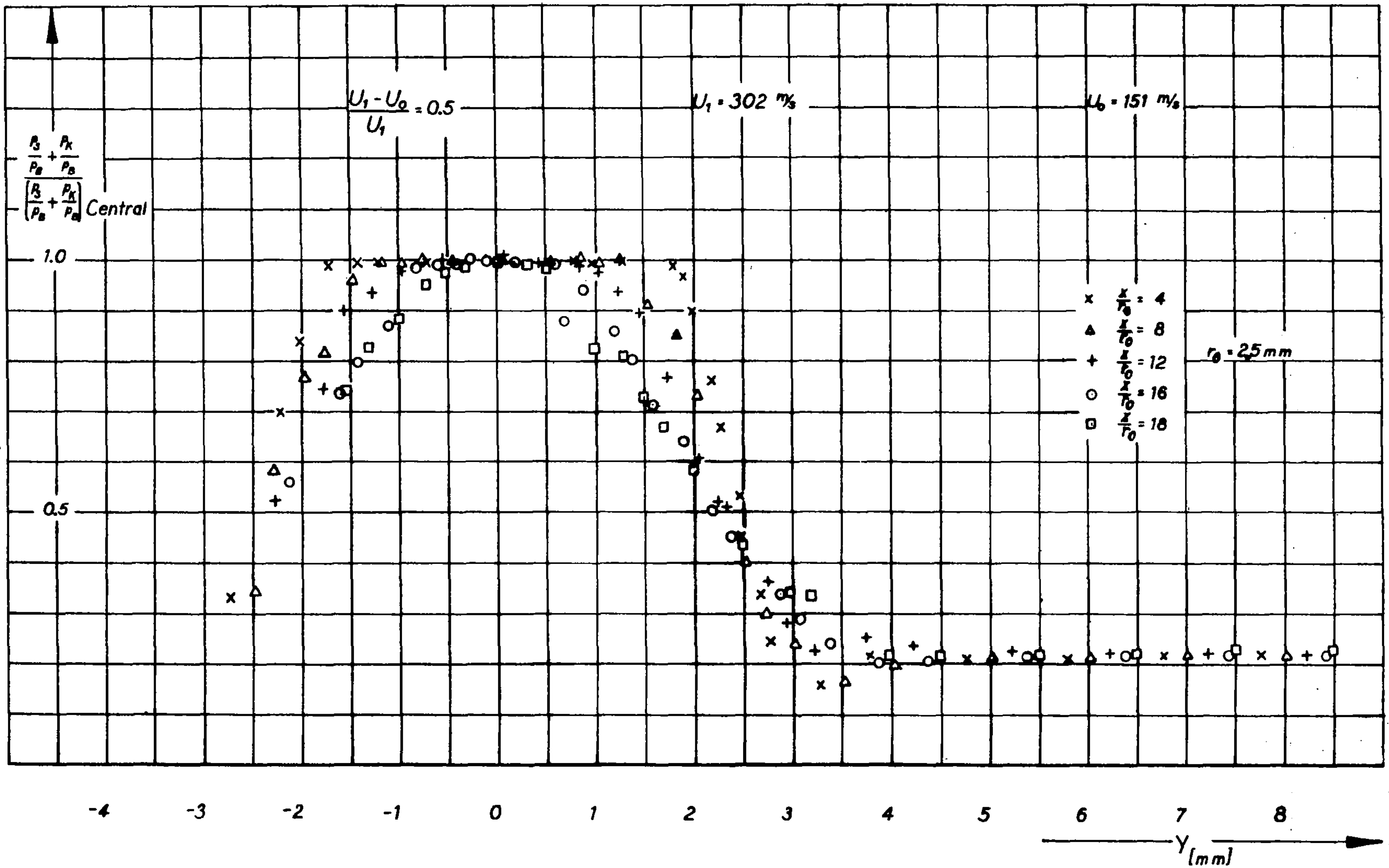


Figure 16.- Auxiliary quantity for calculation of the dimensions of the core region.



$\frac{U_1 - U_0}{U_1} = 0.5$
$U_1 = 302$ m/s
$U_0 = 151$ m/s
Central
1.0
0.5
$\frac{x}{r_0} = 4$
$\frac{x}{r_0} = 8$
$\frac{x}{r_0} = 12$
$\frac{x}{r_0} = 16$
$\frac{x}{r_0} = 18$
$r_0 = 25$ mm
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
Y (mm)

Figure 17.

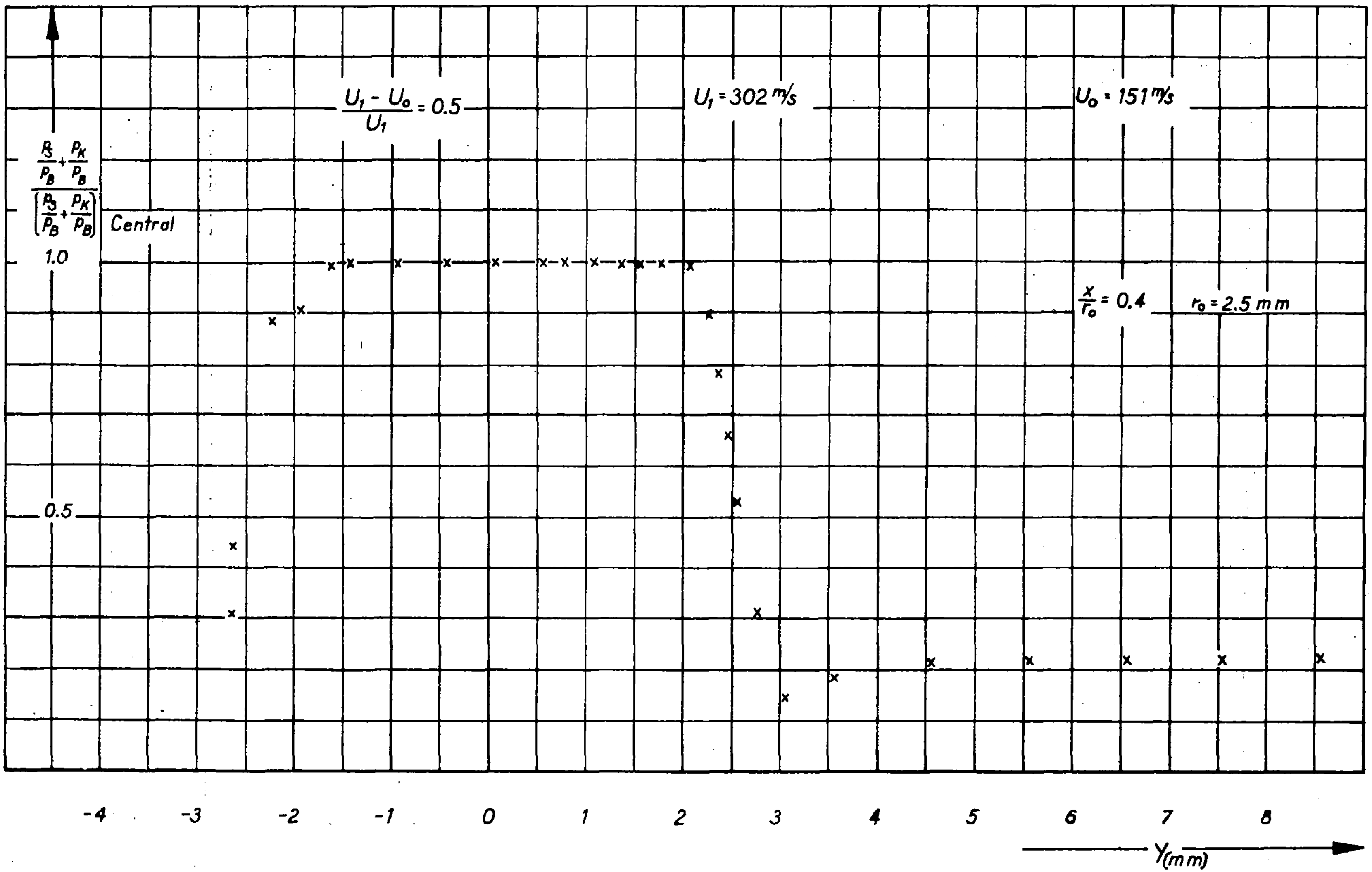

$\frac{U_1 - U_0}{U_1} = 0.5$
$U_1 = 302$ m/s
$U_0 = 151$ m/s
$\frac{\frac{p_S}{p_B}+\frac{p_K}{p_B}}{\left(\frac{p_S}{p_B}+\frac{p_K}{p_B}\right)_{Central}}$
1.0
0.5
$\frac{x}{r_0} = 0.4$
$r_0 = 2.5$ mm
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
Y(mm)

Figure 18.

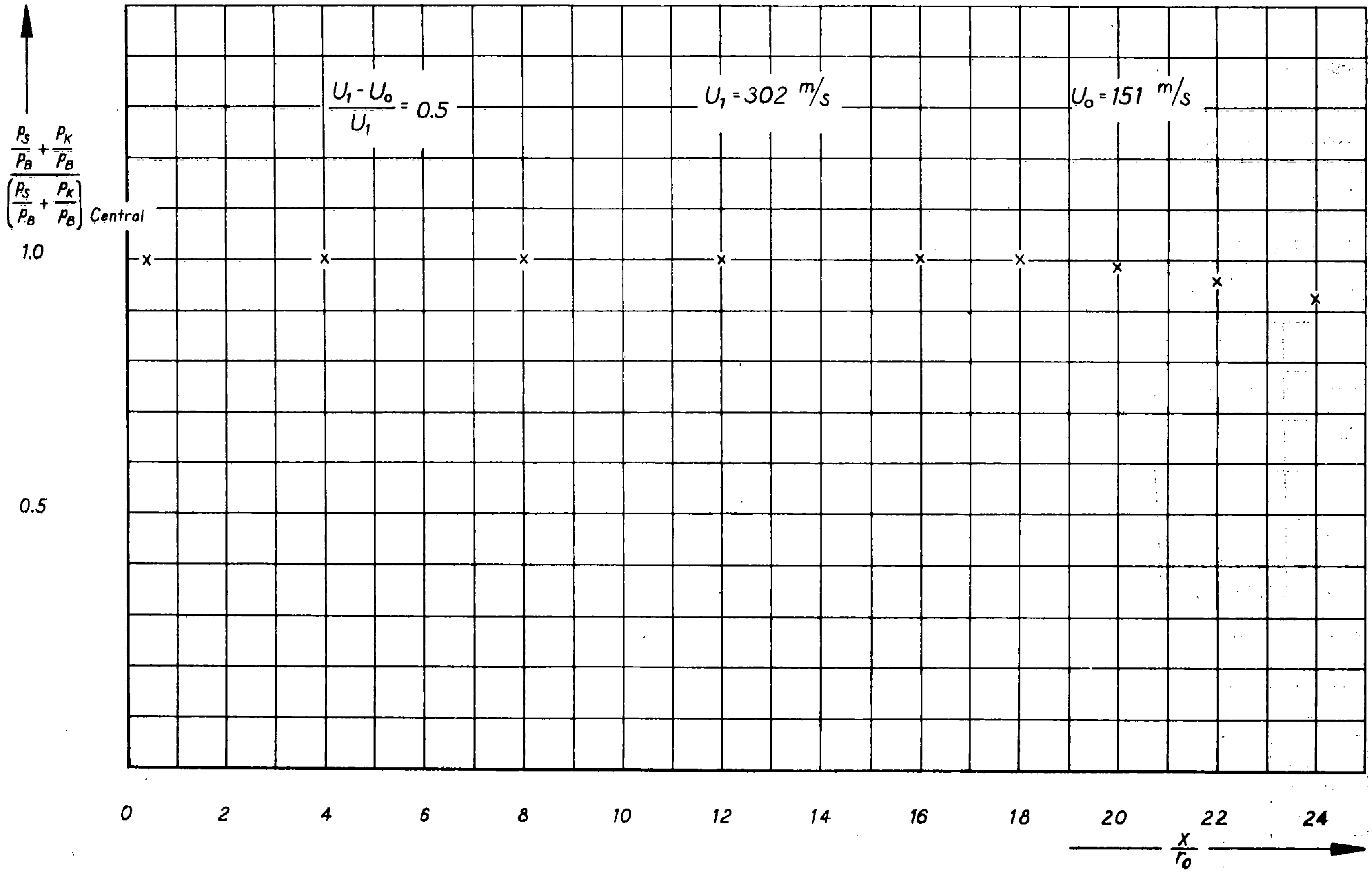
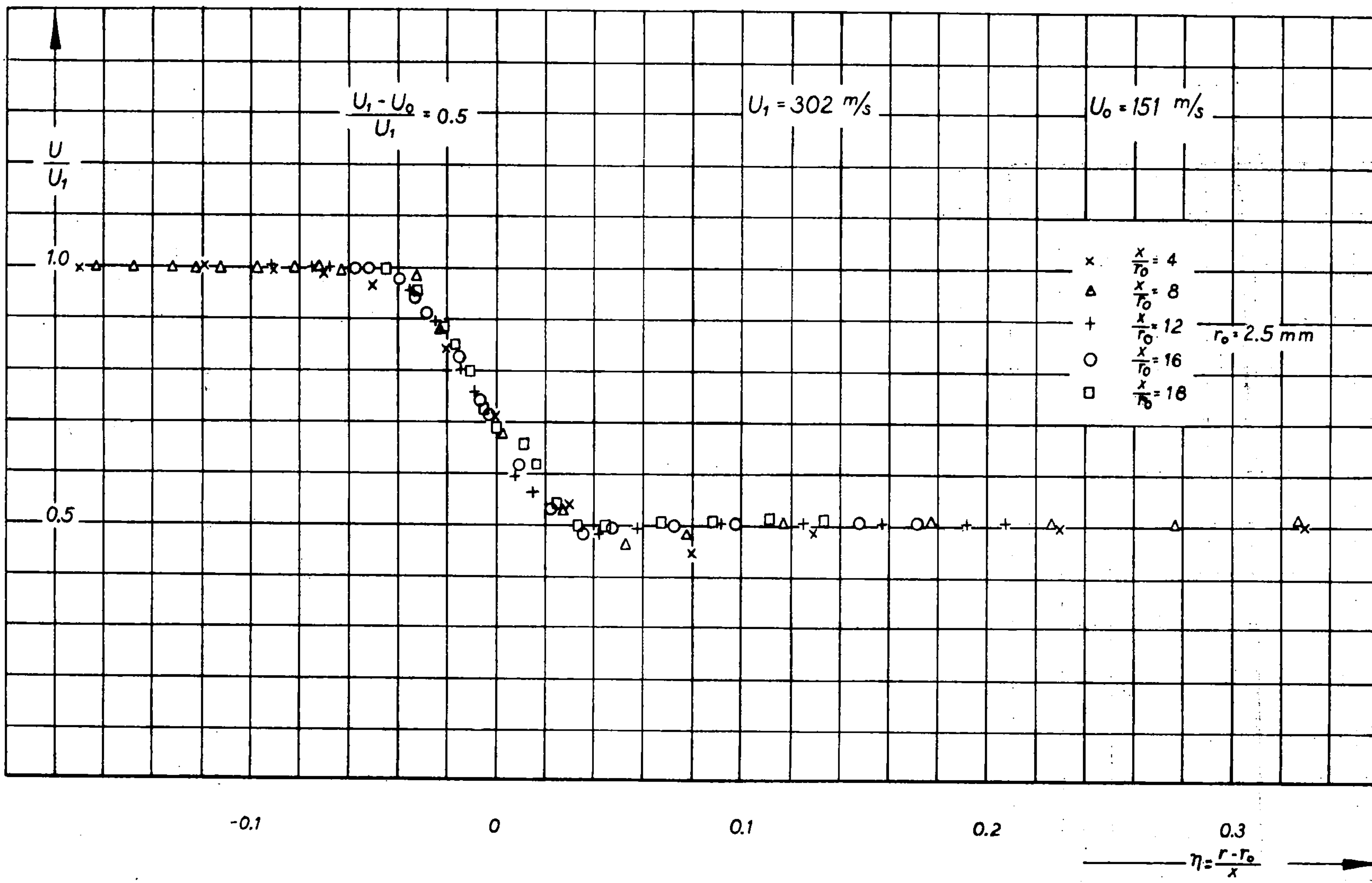

$\frac{\frac{p_s}{p_B}+\frac{p_K}{p_B}}{\left(\frac{p_s}{p_B}+\frac{p_K}{p_B}\right)_{Central}}$
1.0
0.5
$\frac{U_1-U_0}{U_1}=0.5$
$U_1=302\ m/s$
$U_0=151\ m/s$
0
2
4
6
8
10
12
14
16
18
20
22
24
$\frac{x}{r_0}$

Figure 19.



U/U1
1.0
0.5
(U1 - U0)/U1 = 0.5
U1 = 302 m/s
U0 = 151 m/s
x/r0 = 4
x/r0 = 8
x/r0 = 12
x/r0 = 16
x/r0 = 18
r0 = 2.5 mm
-0.1
0
0.1
0.2
0.3
η = (r - r0)/x

Figure 20.

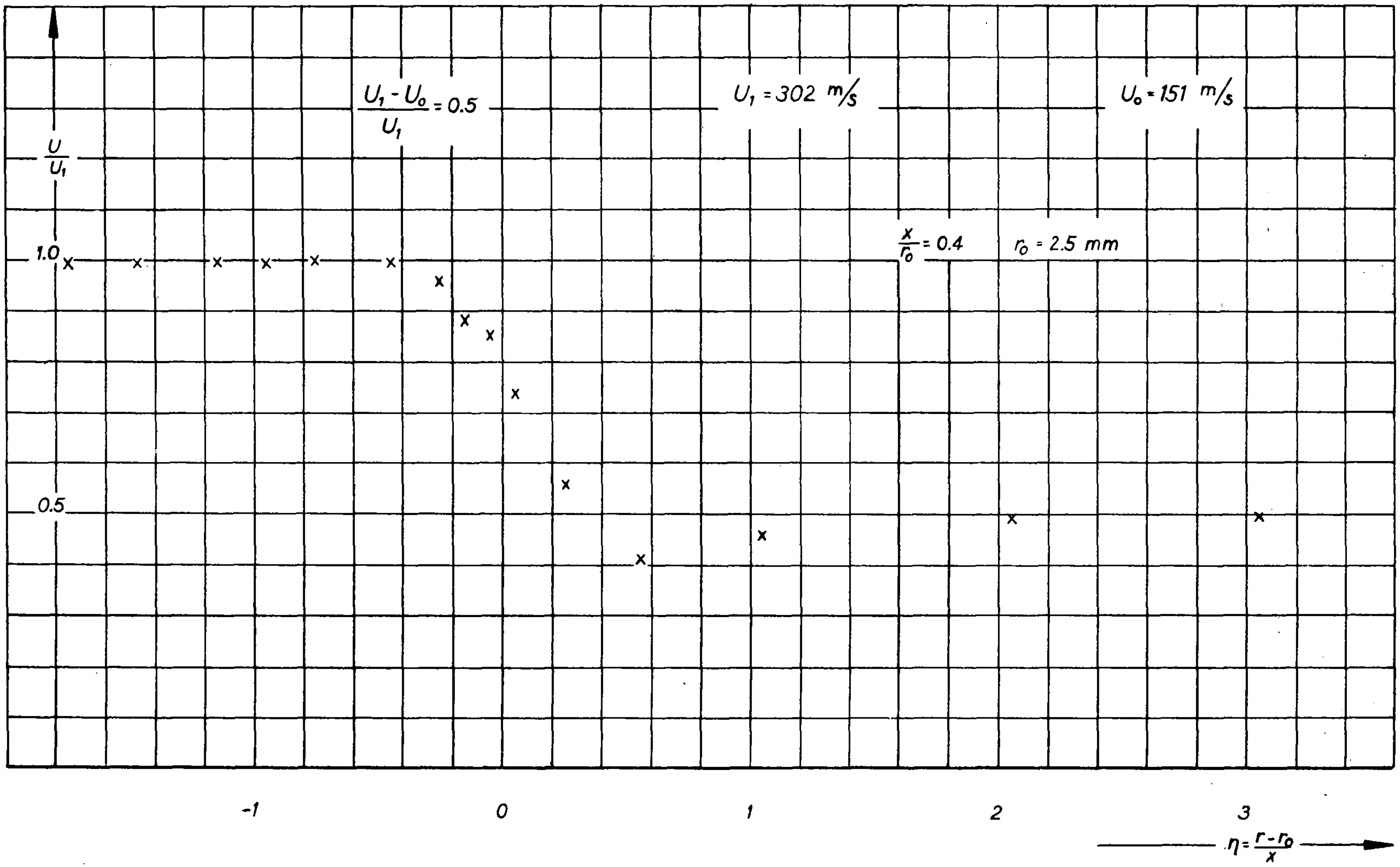

$\frac{U_1 - U_0}{U_1} = 0.5$
$U_1 = 302$ m/s
$U_0 = 151$ m/s
$\frac{U}{U_1}$
1.0
0.5
$\frac{x}{r_0} = 0.4$
$r_0 = 2.5$ mm
-1
0
1
2
3
$\eta = \frac{r - r_0}{x}$

Figure 21.

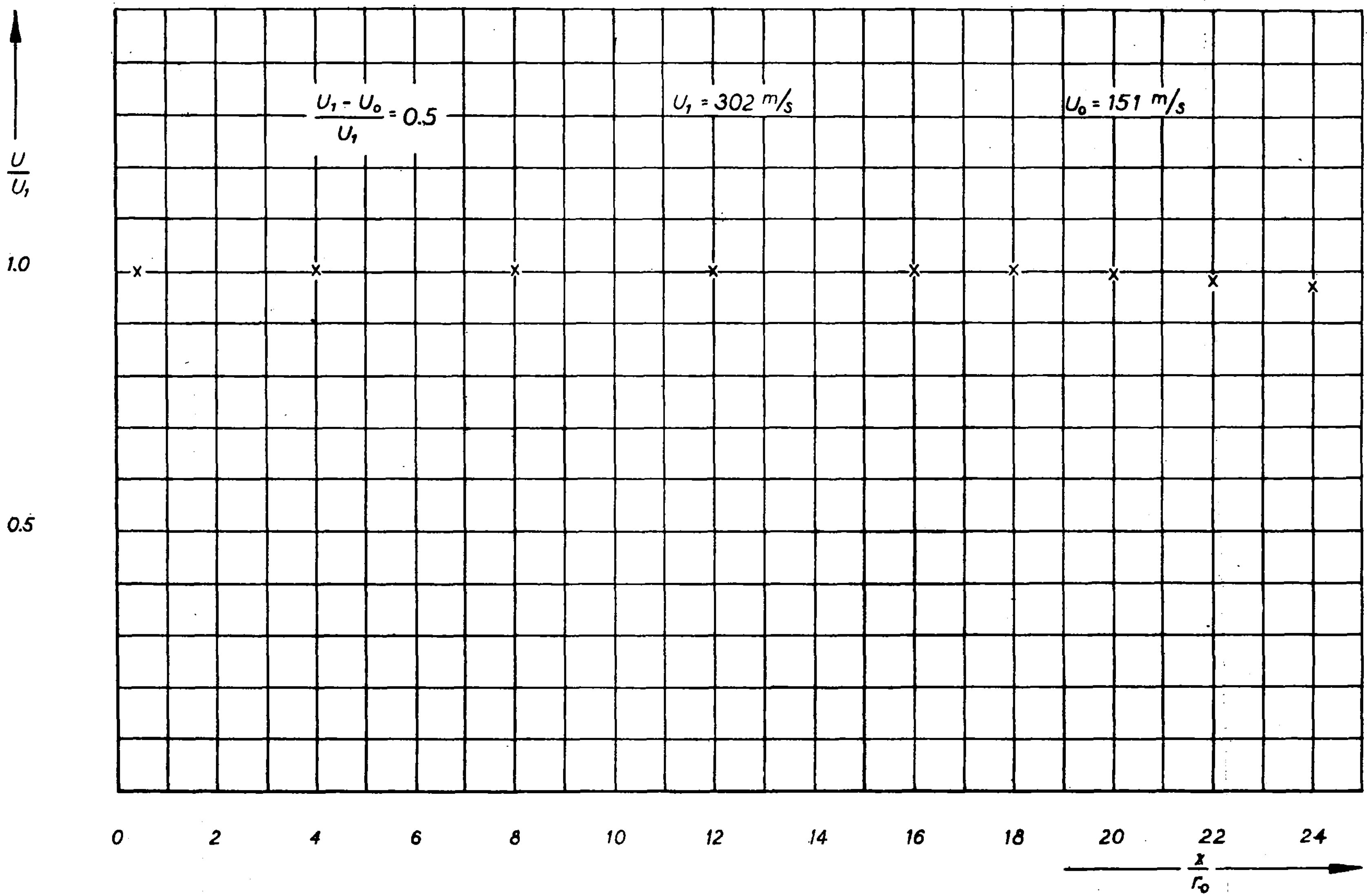

$\frac{U_1 - U_0}{U_1} = 0.5$
$U_1 = 302\ m/s$
$U_0 = 151\ m/s$
$\frac{U}{U_1}$
1.0
0.5
0
2
4
6
8
10
12
14
16
18
20
22
24
$\frac{x}{r_0}$

Figure 22.

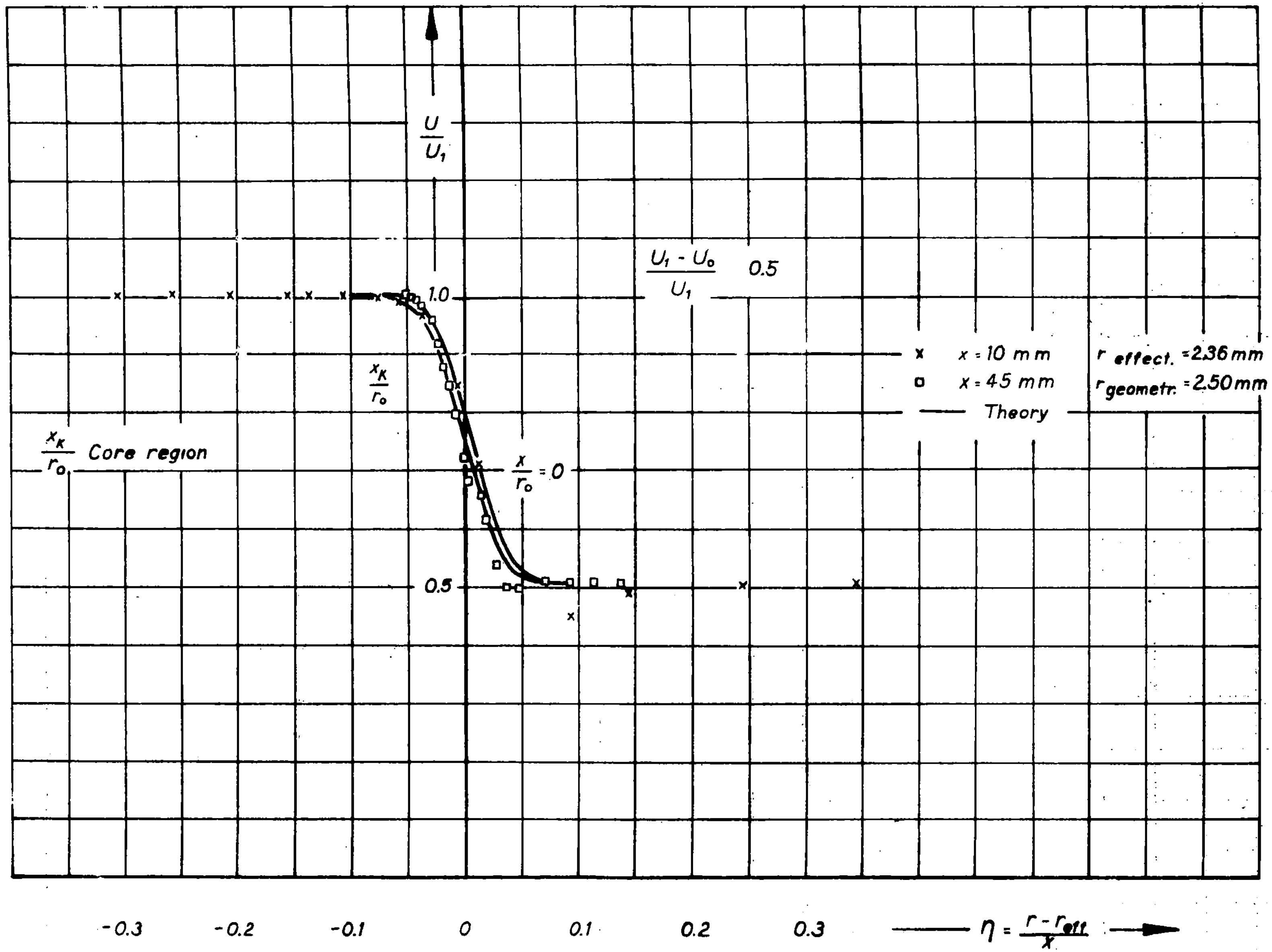


Figure 23.- Comparison between theory and experiment.